This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at http://books.google.com/

4. Nov. 11.

Nov. 12. 1719.

IMPRIMATUR.

Is. Newton, *P. R. S.*

Geometria Organica:

SIVE

DESCRIPTIO

LINEARUM CURVARUM

UNIVERSALIS.

AUCTORE

COLINO MAC LAURIN, *Matheſeos in Collegio Novo* Abredonenſi *Profeſſore, & Reg. Soc. Soc.*

BIBLIOTHECA
MAY 1880
BODLEIANA

LONDINI:

Impenſis GUL. & JOH. INNYS, Regiæ Societatis Typographorum in Areâ Occidentali D. Pauli. MDCCXX.

183. e. 56.

PRÆFATIO.

CUM recondita ac ſublimis Philoſophia, quam mortalibus his tandem ſæculis indulſit benigniſſimum Numen, Geometriæ promotæ accepta referenda ſit; atque Incrementa egregia, quæ plures aliæ Artes ac Diſciplinæ humano generi apprime utiles noſtra ætate ceperunt, ex eodem fonte promanarunt; novos quoſcunque in hac Scientia proceſſus non eſſe contemnendos, colendamque in primis eam eſſe ſtudioſis veritatis indagatoribus, nemo cordatus non fatebitur. Nec varia Geometrarum ad Matheſin puram promovendam Conamina quiſquam vituperabit (etſi fortaſſis primo intuitu non innoteſcat hujuſmodi inventorum utilitas) qui meminerit quam multa, quæ prima fronte parum proficua viſa ſunt, dein temporis decurſu, aliorum induſtria, ad uſus eximios accommodata ſint.

Neque hoc tantum Nomine Geometria ſtudioſe excolenda videtur, quod magnas utilitates afferat, & Ornamento ſit; verum etiam quod innata quadam Pulchritudine plurimum alliciat; imo ei non abſimilem, quam ex humanioribus diſciplinis percipimus voluptatem, in animis noſtris ingeneret. Etenim cuivis, varia vitæ oblectamenta conſideranti, patebit ea ex perceptione Proportionis, in qua omnis ſita eſt Pulchritudo, potiſſimum enaſci; atque animos noſtros ita à natura eſſe compa-

ratos,

ratos, ut rerum perſæpe vulgarium, pæne dixi inanium, Harmonia ſeu Proportione nuda haud parum delectentur. Quid quæſo tantopere juvat in Muſica? Non ſoni, ſed ſonorum harmonia, ex commenſurabili Vibrationum æquabilium proportione aſſurgens; cum ſoni ipſi ſeorſum editi exiguam aut nullam animo pariant delectationem. In omni ſcribendi genere (ne ſingula percurram) poſt ipſam veritatem, juſta rerum diſpoſitio & methodus Auctoribus curæ, & Lectoribus voluptati ſunt. Ipſa vero illa veritas quid eſt aliud, quam, ſive immutabilium idearum, ſive fluxarum rerum, mutuus quidam ordo & habitus, in intimis Naturæ penetralibus delitesſcens; ſedula opera & ſtudio indagandus. Univerſalem porro hujus Proportionis vim penitus animo perſenſiſſe videntur primarii inter veteres; qui etiam ipſam Virtutem in debita affectuum Proportione, & juſto Tono, collocarunt; imo & nonnunquam pronuntiare non dubitarunt, pravam omnem electionem ex ejus Scientiæ defectu oriri, quæ docet Artem menſurandi; *ſcil. infinita cum finitis; magna cum parvis, perfecta cum imperfectis, ſeu bona cum malis comparandi.*

Quorſum autem hæc; niſi ut aliquatenus intelligamus unde tanta inſit amœnitas Theoriis illis, quas ſcientiæ omnes, & ſpeciatim Mathematicæ, intellectui exhibent; & ut ſua puræ Geometriæ conſtet laus & commendatio? Philoſophiæ quidem Mathematicæ præſtantia apud omnes hodie in confeſſo eſt: Quippe quæ animum ad grandiloquam cœlorum facundiam advertendam evocat, univerſique operis formam ſpecioſiſſimam oculis quaſi ſubjicit. Cujus ope, ſive corporis mundani cujuſvis ſeorſum ſumpti motus & Phænomena exquiramus; ſive integri Syſtematis, quatenus adhuc innotuit, diſpoſitionem & ſtructuram contemplemur; ubique ſummum rerum ordinem & harmoniam conſummatiſſimam deprehendentes, ſuaviſſima tanti operis, magnique ejus Auctoris, admiratione non poſſumus non perfundi.

Sed Matheſis pura, quæ partem hujus gloriæ, quanta quanta eſt, longe præcipuam ſibi pro jure ſuo vendicat; licet tam obvia nos voluptate non percellat; menti tamen conceptus ſuppeditat pulcherrimos; quæ Harmoniam & Proportionem ubique deſiderat. His contemplationibus quicunque deditus eſt, ſive unius alicujus accuratæ Lineæ aut Figuræ varias proprietates invicem conſentientes; ſive integram aliquam Linearum

& Fi-

& Figurarum ſpeciem, aut ſpecierum Syſtema, perpendat; aut varia illa ſyſtemata univerſalia conferat; is proportionis omnis, atque adeo pulchritudinis, unam infinitis modis variegatam ideam animos perluſtrat; cum omnis harmoniæ & proportionis Lex aut Regula poſſibilis, curvam habere poſſit ſui obſervatricem. Mentis interea vires hujuſmodi exercitiis mirum in modum augentur & confirmantur. Dum autem, Geometriæ puræ auxilio, tum præſentem Mundi fabricam, tum etiam quæ futura fuiſſent illius ſecundum alias quaſcunque Leges efformati Phænomena, inveſtigamus; indicia undequaque prodeunt luculentiſſima ſapientiſſimi Numinis, ſupremi univerſorum Domini; cujus opus Auctore ſuo dignum, quantum nobis adeo exiguæ ejus extenſionis & durationis partis ſpectatoribus aſſequi licet, omni ex parte deprehendimus. Hæc in eum finem dicta ſunto, ut pateat non inutiles nec injucundas eſſe appellandas illas ipſas puræ Geometriæ Theorias, quæ ad quam vitæ partem perficiendam pertineant non ſtatim apparet; & quarum penitior indagatio ſummorum Virorum exemplo eſt perpetuo comprobata.

Geometria veterum ad Figuras duntaxat, quæ Lineis rectis vel curvis primi generis circumſcribuntur, pertingebat. Recentiores vero, Linearum ordines infinitos in Geometriam receperunt; & æquationibus, ordinatas & abſciſſas curvarum involventibus, definierunt. Nemo tamen, ante Illuſtr. Newtonum, *univerſalem aggreſſus eſt deſcriptionem organicam Linearum Ordinis ſecundo altioris. Hujus methodus commodiſſimam ſuppeditat viam Lineas tertii Ordinis puncto duplice ditatas mechanice deſcribendi; nonnullas etiam Lineas altiorum Ordinum complectitur. His veſtigiis inſiſtentes, Methodum univerſaliſſimam in ſequenti Tractatu tradimus; qua curvæ cujuſcunque Ordinis ſuperioris, motu continuo datorum angulorum ſecundum Lineas rectas, aut etiam ſecundum curvas cujuſcunque inferioris Ordinis, generantur.*

In prima parte, quo pacto Lineæ ſecundi Ordinis, ope unius tantum Lineæ rectæ, deſcribi poſſint oſtendimus. Deinde, ex duabus Lineis rectis, Linearum tertii Ordinis puncto duplice donatarum Geneſin deduximus. Tum Lineas hujus Ordinis, puncto duplice deſtitutas, & quarti, trium Rectarum ope, deſcripſimus. Denique methodum curvas quaſcunque, ope plurium Rectarum, deſcribendi univerſaliſſime demonſtravimus; &

nonnullarum

nonnullarum, etiam vigeſimi quarti Ordinis, deſcriptionem nonnunquam ſex Rectarum ope perfici poſſe, exempli cauſa, oſtendimus.

In ſecunda parte, curvas omnium ſuperiorum Ordinum, ope inferiorum quorumcunque, pariter deſcribi poſſe demonſtravimus. Epicycloides quaſvis, quæ Curvarum revolutione ſuper ſeipſas tanquam Baſes generantur, univerſaliter tractavimus. Expeditam, ad bene multas ſeries infinitas harum Epicycloidum dimetiendas, viam aperuimus. Dein motum corporum, curvas datas in mediis quibuſcunque deſcribentium, & ipſorum mediorum Reſiſtentiam ac Denſitatem parvo admodum labore inveſtigandi methodum expoſuimus. Tandem, qua ratione curvæ Geometricæ dati Ordinis per data puncta duci poſſint, in omnibus caſibus quos adhuc accurate perpendere licuit, oſtendimus.

In primæ Partis Propoſitionibus pleriſque, non ſolum ad quem Ordinem curvæ deſcriptæ pertineant demonſtravimus; ſed etiam quo pacto illarum æquationes Algebraicæ obtineri poſſint oſtendimus. Lectores autem monitos velim, varia admodum, pro variis Rectarum datarum Poſitionibus, & pro varia angulorum datorum ſpecie, exurgere in illis Æquationibus quantitatum ſigna; quæ in aliquibus tantum caſibus deſignare operæ pretium fuit.

In Sectione ultima Partis primæ poſtulavimus, punctorum in quibus Lineæ duæ quæcunque inter ſe occurrunt numerum maximum æqualem ſemper eſſe facto ex numeris, qui ambarum Ordines exprimunt. Cum vero res demonſtratione indigere videretur; eam, quantum adhuc fas fuit, ad omnes caſus accommodatam, ad calcem ſubjunximus poſthac fortaſſis ſuis numeris abſolvendam. Quod, vero hæc qualiacunque rudiori quam optaſſemus manu pertractata in lucem edere coacti ſimus, exinde accidit; quod otium & prœli facultas, defuerunt.

Linearum

VIRO ILLUSTRISSIMO

D. *ISAACO NEWTON*, Eq. Aur.

Regalis Societatis PRÆSIDI.

ETSI non deerant, Vir ampliſſime, Cauſæ variæ, quæ hanc audaciam, magno tuo Nomini Lucubratiunculas haſce inſcribendi, cohibere poſſent; vicit tamen eas omnes ſingularis tua in me meoſque conatus qualeſcunque Benevolentia; cui rependendæ quidem nunquam par ero, agnoſcendæ certe nulla in re deeſſe debeo. Nec erat cui pari jure dicari poſſent hæc Theoremata: Te enim prælucente indagata ſunt; Tuis etiam Auſpiciis nunc in Lucem edita. Nec quicquam Matheſeos Cultoribus prius eſſe debet, quam ut Magno NEWTONO, quavis occaſione oblata, gratum animi ſenſum publice profiteantur. Si enim prior iſtarum Scientiarum tenuitas, cum eo ad quod à Te ſubvectæ ſunt Dignitatis & Splendoris Faſtigio, conferatur; næ Tu ad id natus videris,

ut ſpecimen exhiberetur, quantam Ratio humana, veritatibus ſublimioribus indagandis, operibuſque naturæ perſcrutandis incumbens, profundæ rerum Caligini Lucem diemque, inferre poſſet.

Ante annos non ita multos Geometria, cujus ditiones in immenſum proferri queunt, etſi a Magnis Viris plurimum aucta, intra anguſtos tamen limites continebatur. Philoſophia vero Naturalis, Geometriæ Præſidiis nimium deſtituta, & proinde laſcivientis cujuſque ingenii petulantiæ obnoxia, inſulſis & inter ſe pugnantibus Philoſophorum ſomniis diu multumque vexabatur. Vanarum Hypotheſeων, Commentorumque omni Rationis Patrocinio carentium, nec finis erat nec modus. Immanes de rerum natura Fabulas, à Mentis Rectricis ſapientia maxime alienas, comminiſci, aliiſque ſuperciliose obtrudere, id demum erat philoſophari. Non Religio erat mirificam opificii divini Harmoniam figmentis diſcordibus corrumpere, venuſtam Naturæ faciem indigniſſimo fuco deformare, ordinem exquiſitiſſimum perturbare. Ad ſingula Naturæ Phænomena ſolvenda ſingulæ fere fingebantur cauſæ; quarum porro ipſarum explicatio, quam Phænomenων illorum, longe erat difficilior. Hujuſmodi erant Materiæ ſubtilis Mæandri; Vorticum & Epicyclorum ambages; aliaque id genus plurima.

Philoſophia Naturalis, tot malis laborans, diu vindicem deſiderabat, cujus auſpiciis è Fabularum Caligine emergens, Nativo ſuo decori & ſplendori reſtitueretur. Cl. quidem *Verulamius*, Britanniæ perenne Decus, Philoſophos præmonuerat ut Experimentorum

ſuffragia

ſuffragia ſedulo opperirentur, neque ea præmaturis Hypotheſibus anteverterent; iis vero Veritatibus Univerſalibus indagandis præcipue incumberent, ex quibus, cum Experimentorum & Phænomenωn obſervatione apte conſociatis, vera Lux Philoſophica eliceretur. Verum deerant quibus Ingenium, Animus, Patientia, ad tantos Labores ſuppeditarent. Tu tandem unus, tam ſtupendo Operi par, innumeras Veritates reconditiſſimas Philoſophiæ Naturali mirifice inſervientes indagaſti; indagatas fœliciſſime ad Phænomena Naturæ explicanda adhibuiſti; atque ex multiplici ſubtiliſſimorum Experimentorum penu, magis magiſque confirmaſti; nec cujuſquam ante Te veſtigiis inſiſtens, nec à quoquam in poſterum æquandus.

Geometria demum ultra priſtinos fines, incredibili Tua Perſpicacitate & Ingenii vi, longe lateque exporrecta eſt; recondita præſertim Curvarum Theoria mirum in modum aucta; quantitatum fluentium Algorithmus inventus, & ad ingentia Naturæ Opera enucleanda ſummo Judicio adhibitus; Philoſophia Naturalis in Libertatem aſſerta, variiſque Errorum monſtris purgata; materia ſubtilis exploſa; eliminata prorſus è Cœlis Vorticum & Epicyclorum ſupellex: Variæ & perplexæ admodum Lunaris motus inæqualitates calculo & certis Legibus ſubjectæ; Cometarum orbitæ per remotiſſima Cœli ſpatia pertinaciter inveſtigatæ; ſubtiliſſima Luminis Phænomena ex intimis Naturæ penetralibus Arte mirifica eruta; Corporum terreſtrium Gravitas, & cœleſtium Motus, cum innumeris aliis Naturæ Phænomenis, ad eandem ſimplicem Cauſam

 reducta;

reducta; denique, pulſis Fabulis, vera operum divinorum Majeſtas, Ordo, Harmonia, Omnium Contemplationi ſimul & Admirationi commendata. Quæ omnia, ſeu Veritatis æterni Triumphi, ſeu Errorum diu graſſantium clades, tanquam monumenta Ære & Marmore perenniora, certatim ad Nominis Tui perpetuitatem poſteris mandandam conſpirant. Cæterum de rebus tam præclaris præſtat ſilere, quam levia dicere. Quocirca, id unum oro, Vir Spectatiſſime, ut hanc, utcunque tenuem, magni tamen Studii voluntatiſque propenſiſſimæ Teſſeram, in bonam partem accipias.

Tui Nominis

Obſervantiſſimus Cultor

Colin Mac Laurin.

Linearum Geometricarum
DESCRIPTIO
Organica Univerſalis.
PARS PRIMA.

Ubi Methodo Univerſali Lineæ omnium Ordinum deſcribuntur ſola datorum Angulorum & Rectarum Ope.

SECTIO I.
De Deſcriptione Curvarum primi Generis ſeu Linearum Ordinis ſecundi.

PROP. I.

Circa duo puncta C & S *in Plano quovis data tanquam Polos moveantur Anguli dati* FCO & KSH; *ducatur concurſus Crurum* CF & SK *per Rectam* AE *in eodem Plano Poſitione datam; atque reliqua interea Crura* CO & SH *Concurſu ſuo* P *deſcribent Curvam primi Generis.*

Jungantur puncta data C & S; atque in Rectam CS demittantur normales PM & QN; ducantur Rectæ PR, QU, PT, QL ita ut Anguli CUQ & CRP æquales ſint ſibi mutuo atque dato FCG, Anguli vero SLQ & STP æquales dato KSD. His poſitis, Angulus RCP æqualis erit Angulo CQU, propterea quod RCP duos rectos conficiat cum Angulis RCQ & QCA, atque CQU duos etiam rectos conficiat cum Angulis RCQ &

QUC dictis RCQ & QCG æqualibus; proinde cum æquales ſint Anguli CUQ & CRP manifeſtum eſt Triangula CQU, CPR eſſe ſimilia; & eadem ratione demonſtratur ſimilia eſſe Triangula SLQ & STP.

Unde erit CR : PR : : QU : CU
atque ST : PT : : QL : SL

Sit CS $= a$, CA $= b$, ſinus Anguli FCO ad ejus Coſinum ut d ad a, Sinus Anguli KSH ad ejus Coſinum ut e ad a, ſinus Anguli CAE ad Coſinum ut c ad a; ſit porro PM $= y$, CM $= x$, QN $= z$, atque erunt RM : PM : : $a : d$. PR : PM : : $\sqrt{d^2 + a^2} : d$. AN : QN : : $a : c$ unde RM $= \frac{ay}{d}$, CR $= x - \frac{ay}{d} = \frac{dx - ay}{d}$, PR $= \frac{y\sqrt{d^2 + a^2}}{d}$,

QU $= \frac{z\sqrt{d^2 + a^2}}{d}$ & CU = CA — AN — NU $= b - \frac{az}{c} - \frac{az}{d}$

$= b - \frac{a \times \overline{d + c} \times z}{dc}$. Cumque ſit CR ad PR ut QU ad CU patet, eſſe

$\frac{dx - ay}{d} : \frac{y\sqrt{d^2 + a^2}}{d} :: \frac{z\sqrt{d^2 + a^2}}{d} : \frac{bdc - a \times \overline{d + c} \times z}{dc}$; unde

QN $= z = \frac{bc \times \overline{dx - ay}}{\overline{dc - a^2} \times y + a \times \overline{d + c} \times x}$.

Porro ſimili Ratiocinio prodit ST $= a - x - \frac{ay}{e}$, PT $= \frac{y\sqrt{a^2 + e^2}}{e}$,

QL $= \frac{z\sqrt{a^2 + e^2}}{e}$ & SL = AN — AS — NL $= a - b + \frac{a \times \overline{e - c} \times z}{ec}$.

Eſt vero ST ad PT ut QL ad SL; ſeu $a - x - \frac{ay}{e} : \frac{y\sqrt{a^2 + e^2}}{e} ::$

$\frac{z\sqrt{a^2 + e^2}}{e} : \frac{\overline{a - b} \times ec + a \times \overline{e - c} \times z}{ec}$ & proinde QN $= z =$

$\frac{\overline{a - b} \times c \times \overline{ae - ex - ay}}{\overline{ec + a^2} \times y + a \times \overline{e - c} \times x + a^2 \times \overline{c - e}}$. Æquentur duo reperti Valores Rectæ QN, & Denominatorem unius ducendo in Numeratorem alterius, prodibit Æquatio Curvæ deſcriptæ ad Axem C S

$$\left.\begin{matrix} \overline{a - b} \times ce \\ + \overline{ae - bc} \times d \end{matrix}\right\} x^2 \quad \left.\begin{matrix} + a^2 \times \overline{d + c - e} \\ + dce \end{matrix}\right\} yx \quad \left.\begin{matrix} + c \times \overline{-a^2 + cd} \\ - bc \times \overline{e + d} \end{matrix}\right\} y^2$$

$$\left.\begin{matrix} + abc \times \overline{d + e} \\ - a^2 e \times \overline{d + c} \end{matrix}\right\} x \quad \left.\begin{matrix} - bc \times \overline{a^2 - ed} \\ - ae \times \overline{dc - a^2} \end{matrix}\right\} y = 0$$

Cum

Cum vero in hac Æquatione variabiles Quantitates x & y ad duas tantum Dimenſiones aſcendant, manifeſtum eſt curvam deſcribi primi Generis ſeu Lineam ſecundi Ordinis; cujus Interſectiones cum Linea recta poſſunt eſſe duæ.

Corol. I. Curva hac Propoſitione deſcripta tranſit per data puncta C & S; nam ſi $y = o$ Æquatio Curvæ evadet $\overline{a-b}\times ce+\overline{ae-bc}\times d\times x^2 + abc\times\overline{d+e} - a^2 e\times\overline{d+c}\times x = o$. Proinde cum $\overline{a-b}\times ce+\overline{ae-bc}\times d = ae\times\overline{d+c} - bc\times\overline{d+e}$ erit $x^2 - ax = o$ adeoque vel $x = o$ vel $x = a$; in priore caſu curva tranſit per C in poſteriore per S. Idem ex Conſtructione patet immediate; quippe dum SH coincidit cum SC Curva tranſit per C; & CO evadit Linea Curvam in dato puncto C contingens; Curva vero tranſit per S dum CO coincidit cum CS, & crus SH Curvam tunc in puncto S contingit; quod ſatis eſt manifeſtum. Puncta C & S ſunt ſimplicia, nec Linea quævis Ordinis tertio inferioris punctum habet duplex.

Corol. II. Hinc ſi dati Anguli HSX, OCX moveantur circa puncta C & S ut Polos, & crurum SH, CO concurſus P ducatur per Lineam ſecundi Ordinis, ſeu ſectionem Conicam, per data puncta C & S tranſeuntem; concurſus X crurum CX & SX percurret aliam ſectionem Conicam; nam dabuntur Anguli QSX & QCX; adeoque cum Q percurrat Rectam, punctum X deſcribet Lineam Ordinis ſecundi ſeu ſectionem Conicam per hanc Propoſitionem.

Theorema hoc primus demonſtravit D. *Neutonus* egregio Ratiocinio ad morem veterum *Lemmate* 21°, *Lib.* 1^mi^ *Principiorum*; idem quoque analytice inveſtigavit in *Arithmetica Univerſali*; Analyſin vero diverſam hac Propoſitione exhibuimus, quoniam noſtra in ſequentibus calculis uſus erit non exigui & æquationem ad Ordinatas axi normales exhibeat.

Prop. II. Problema.

Determinare Curvarum Aſſymptotos atque Species.

SUper datâ Rectâ CS *(Fig.* 2.*)* deſcribatur ſegmentum circuli cui inſcribi poſſit Angulorum datorum QCP, QSP ſupplementum ad quatuor rectos. Si Recta data AE huic circulo bis occurrat deſcribetur Hyperbola Conica; ſi Recta AE circulum contingat deſcribetur Parabola; ſi recta AE tota ponatur extra Circulum curva deſcripta erit Ellypſis.

Imprimis occurrat Recta AE Circulo in punctis duobus Q & N; cumque SK pervenerit ad poſitionem SQ vel SN manifeſtum eſt crura CO & SH evadere parallela, eorumque concurſum P abire in utroque caſu in infinitum; quippe Quadrilateri QSCP quatuor anguli æquales ſunt quatuor rectis, adeoque cum CQS, QCP, QSP ex hypotheſi, totidem conficiant rectos, manifeſtum eſt angulum CPS evaneſcere, & crura CP & SP concurrere ad Diſtantiam infinitam. Similiter cum SK pervenerit ad poſitionem SN, erunt $C\pi$ & $S\pi$ parallelæ, atque puncta P & π eundo ac redeundo in infinitum, quatuor deſcribent crura infinita ad duas Aſſymptotos rectis CP & $C\pi$ parallelas; quæ ſe mutuo ſecant in Centro Figuræ, angulo æquali QCN ſeu QSN.

Ut Poſitio Aſſymptoton prodeat, ſit q punctum ipſo Q quamproximum & anguli qSp, qCp æquales fiant datis QSP & QCP, atque erit p punctum ipſo P quamproximum; unde ſi pB ducatur parallela rectæ CP, erit Bp curvæ Aſſymptotos. Eſt vero angulus $SpB = pSP = QSq$ & anguli quoque BpC, pCP, QCq æquales; ſed patet angulos evaneſcentes BpS & BpC eſſe in ratione ſubtenſarum BS & BC, & proinde erit BS ad BC ut QSq ad QCq. Sed anguli quicunque ſunt ut arcus ſubtendentes directe & arcuum radii reciproce, adeoque QSq erit ad QCq ut $Qu \times CQ$ ad $qe \times SQ$ ſive (cum Qu ſit ad Qq ut perpendiculum SF S in AE demiſſum ad SQ; & ſit qe ad Qq ut CG perpendiculum ex C in AE demiſſum ad CQ) ut $SF \times CQ^2$ ad $CG \times SQ^2$. Proinde BS erit ad BC ut $AS \times CQ^2$ ad $AC \times SQ^2$ & BC ad CS ut $AC \times SQ^2$ ad $AC \times SQ^2 \pm AS \times CQ^2$; unde ſi anguli QgS, QTC ſint æquales ſibi mutuo & dato angulo CQS, erit BC ad CS ut $AC \times Sg$ ad $AC \times Sg \pm AS \times CT$. Proinde dabitur BC, & angulus CBp datur, adeoque poſitione determinatur Aſſymptotos parallela rectæ CP; ſimiliter plane determinari poteſt altera Aſſymptotos parallela rectæ $S\pi$; duarum vero concurſu determinatur centrum figuræ.

Ex his conſequitur 1° Curvam deſcriptam fore Hyperbolam Conicam, quoties recta AE circulum SDC in duobus punctis ſecat; & ſi illa duo puncta ſint Q & N, patet ex dictis, quo pacto duæ Aſſymptoti determinari poſſunt & ſpecies proinde curvæ inveſtigari. 2°. Si recta AE circulum contingat, coincidunt Q & N cum D, curvæ Aſſymptoti abibunt in infinitum & curva deſcripta erit Parabola Conica. Quippe angulus ADS æqualis erit angulo DCS per *Prop.* 32. Lib. 3″. *Elem.* & proinde ſimilia erunt triangula ASD & ADC, adeoque erit AS : AD :: SD : CD, ſed per *Prop.* 36. ejuſdem Libri eſt AS : AD :: AD : AC & proinde AS : AC :: AS^2 : AD^2 :: SD^2 : CD^2. Sed prius demonſtravimus eſſe BS : BC :: $AS \times CQ^2$: $AC \times SQ^2$ :: (in hoc caſu)

caſu) AS × CD² : AC × SD² : : AS × AC : AC × AS unde BC : CS : : AC×AS : AC × AS — AS×AC : : AC×AS : *o i. e.* BS evadit infinite major quam CS; unde Aſſymptotos abit in infinitum & crura curvæ evadunt Parabolica. Proinde in hoc caſu deſcribitur Parabola cujus duæ diametri erunt rectæ SP & CP (SK perveniente ad poſitionem SD) earumque vertices C & S, tangentes vero ad eos vertices C & S per Corol. I. Prop. I. atque ex his Parabola facile determinantur. 3°. Quod ſi Recta data AE Circulo neque occurrat, nec eum contingat, ſed tota extra circulum cadat, deſcribetur Ellypſis; quippe in hoc caſu rectæ CP & SP, quarum concurſu Curva determinatur, nunquam poſſunt evadere parallelæ; & proinde curva non abit in infinitum ſed in ſe redit.

Corol. I. Axes Figurarum non minus quam Aſſymptoti ſic facile geometrice determinantur; ſit O (*Fig.* 3.) centrum circuli SQC, & ex O in Rectam AE cadat Linea perpendicularis OX Circulo occurrens in punctis Y & G atque rectæ AE in X; & ſi anguli HSK crus SK ducatur ad poſitionem SY, alterum crus SH evadet parallelum axi majori & perpendiculare axi minori figuræ in poſitione nim. Sz; nam YN = YQ & angulus YSN = YSQ = $\frac{1}{2}$ NSQ = PSz; cumque NSQ æqualis ſit angulo quo Aſſymptoti ſe mutuo interſecant, erit PSz dimidium illius anguli, adeoque Sz parallela axi majori quæ Aſſymptotωn angulum biſecat. Sed ex Conicis conſtat axem majorem eſſe ad minorem ut co-ſinus dimidii anguli Aſſymptotωn ad ſinum dimidii ejuſdem anguli; eſt vero YSN, qui æqualis eſt dimidio anguli Aſſymptotωn etiam æqualis angulo YGN; adeoque erit GX ad XN ut Axis major ad minorem; ſed GX : XN : : XN : XY & GX : XY : : GX² : XN²: proinde Quadrata Axium erunt ad ſe mutuo ut GX ad XY.

Corol. II. Manente recta AE atque Angulorum QCO & QSK ſumma, manet ſpecies Curvæ; quippe his manentibus, non mutatur angulus NSQ, æqualis angulo in quo Aſſymptoti ſe mutuo ſecant in centro figuræ; adeoque manet ſpecies figuræ.

Corol. III. Si una Aſſymptotωn poſitione determinetur ut B*p*, altera nullo labore dabitur; nam ſi *b* ſit punctum ubi quæſita ſecat rectam CS, erit C*b* = SB, ex natura Hyperbolæ.

Corol. IV. In nullo caſu niſi cum Recta AE abit in infinitum deſcribi poteſt Circulus; quippe ſi punctum deſcribens P Circulum percurreret, cum Curva neceſſario tranſeat per data puncta C & S, angulus CPS eſſet invariabilis, adeoque & CQS; id vero fieri non poteſt dum Q rectam tangit, niſi recta illa ſit ad diſtantiam infinitam.

Corol.

Corol. V. Si anguli FCO, KSH ſint ſibi mutuo ſupplementa ad duos rectos, Aſſymptoti longe facilius determinari poſſunt; quippe evaneſcit angulus CQS in hoc caſu; atque tunc evadunt SP & CP parallelæ, quando punctum Q (*Fig.* 4.) pervenit ad A ubi Recta AE occurrit SC productæ ſi opus eſt, & quando Q abit ad diſtantiam infinitam; cum in utroq; caſu evaneſcat angulus CQS. In priori caſu erit BS : BC : : ($AS \times CQ^2 : AC \times SQ^2$) $AS \times CA^2 : AC \times SA^2$: : AC · AS; & BS : CS : : AC : CS adeoque BS = CA; unde facillime determinatur Aſſymptotos B*p* quæ ad punctum B rectæ CS inſiſtit & Angulum cum eâ recta conſtituit æqualem dato FCO. Altera Aſſymptotos tranſit per A per *Corol.* 3[um]. ſecans rectam CS angulo æquali differentiæ angulorum datorum KSH & CAE; & ex datis duabus Aſſymptotis illis dabitur curva.

Corol. VI. Si anguli ut in ſuperiori *Corolario* ſibi mutuo ſupplementa ſint ad duos rectos, & recta AE occurrat rectæ CS productæ, deſcribetur Hyperbola; ſi recta AE parallela ſit rectæ CS, abibit Aſſymptotos B*p* in infinitum, cum ſemper ſit SB = CA; adeoque curva erit Parabola Conica; Aſt in nullo caſu, ſi angulorum KSH, FCO ſumma ſit æqualis duobus rectis, deſcribetur Ellypſis.

Corol. VII. Quoties recta AE tranſit per circuli SQC centrum angulus NSQ evadit rectus & proinde Aſſymptoti ſe mutuo ſecabunt ad angulos rectos, adeoque Hyperbola deſcribetur Æquilatera.

Corol. VIII. Aliæ etiam ſuppetunt Methodi quibus ſpecies curvarum diſcriminantur, præter eam quam hac Propoſitione explicavimus; eam vero nos adhibuimus quæ in ſequentibus uſus erit frequentioris. Notandum interea crura angulorum quibus curva deſcribitur, conſiderari debere tanquam Rectas infinitas ad datos angulos ſe mutuo ſecantes; alias pars tantum curvæ deſcribetur; & in omni linearum ſuperiorum conſtructione, ope inferiorum quarumcunque, hæ inferiores ſunt totæ aſſumendæ, non earum Arcus quicunque ad integram ſuperiorem ducendam; quod ſemel monuiſſe ſufficiat.

PROP.

PROP. III.

Si in Deſcriptione Prop. primæ angulorum KSH *&* FCO, *quorum motu curva deſcribitur, crura* SH *&* CO *ſimul coincidant cum recta* SC, *punctum deſcribens* P *percurret Lineam rectam non curvam.*

DUcatur CK *(Fig.* 5.*)* Normalis ad AC, & producatur SQ donec ei occurrat in K; ducantur quoque QG & SL parallelæ rectis CS & AE, normali CK occurrentes in G & L. Erit angulus QSC = KQG, & QCO = CQG, & EAC = EQG; adeoque iiſdem retentis ſymbolis ac in Prop. I. erit QG : GK :: $a : e$, QG : GC :: $a : d$; QG : GE :: $a : c$ & GK : GC :: $e : d$ & GC : GE :: $d : c$; unde CK : GC :: $d+e : d$; atque EC : GC :: $d+c : d$, & proinde CK : EC :: $d+e : d+c$: ſed CK $= e$ cum CS ſit æqualis a; atque ut CS : CK :: CA : CE : *i.e.* $a : b :: c :$ CE $= \frac{bc}{a}$ unde conſequitur $e : \frac{bc}{a} :: d+e : d+c$ & $\overline{d+e}\times bc = \overline{d+c}\times ae$. Proinde in æquatione generali Prop. I. evaneſcit terminus $\overline{abc\times\overline{d+e} - a^2e\times\overline{d+c}}\times x$ cumque $\overline{a-b}\times ce + \overline{ae-bc}\times d = \overline{\overline{d+e}\times bc - ae\times\overline{d+c}}$ evaneſcit quoque terminus primus $\overline{\overline{a-b}\times ce + \overline{ae-bc}\times d}\times x^2$ & manebunt tres tantum termini qui dividi poſſunt per y; quo quidem facto, prodit æquatio lineæ unius tantum dimenſionis

$$\left.\begin{array}{l} a\times\overline{-a^2+cd} \\ -bc\times\overline{e+d} \end{array}\right\} y \quad \left.\begin{array}{l} +a^2\times\overline{d+c-e} \\ +dce \end{array}\right\} x \quad -bc\times\overline{a^2-ed} \quad -ae\times\overline{dc-a^2} \quad = 0.$$

Quæ æquatio eſt ad rectam. Proinde quoties anguli CSP, SCP ſimul evaneſcunt, non deſcribetur Linea quævis curva ſed recta.

Corol. I. Recta deſcripta ex æquatione determinari poteſt, ſed facilius hoc pacto: ſumatur CH ad CS ut AS × CQ² ad AC × SQ² — AS × CQ², eritque H punctum ubi recta occurrit datæ CS. Per S, Q & C ducatur Circulus ſecans AE in N, & per punctum H ducatur recta HR angulum CHR conſtituens æqualem angulo NSQ; recta HR ea erit quæ deſcribetur concurſu crurum SH & CO ad modum *Prop. primæ*. Punctum H inveſtigatur ſumendo *q* ipſo puncto Q quamproximum, & inveſtigando *h* ubi concurrunt crura CO & SH cum Q ad *q* pervenerit, cui quamproximum erit ipſum punctum H. Et quoniam cum SK pervenerit ad poſitionem SN abibit P in infinitum, hinc conſtat inclinatio ea rectæ HR ad CH quam explicuimus.

Corol. II. Hinc ſi anguli KSH & FCO ſibi mutuo ſupplementa ſint ad duos rectos, & recta AE angulum conſtituat CAE angulo FCO æqualem, non deſcribetur curva, ſed recta LR, ita poſita, ut CH = SA & angulus EHR æqualis alterutro angulorum FCO & KSH; nam in eo caſu abit Q in infinitum & CQ = SQ; adeoque CH : CS :: AS × CQ² : AC × SQ² — AS × CQ² :: AS : AC — AS :: AS : CS & proinde CH = AS.

Corol. III. Speciatim ſi anguli ſint recti, & AE ſit perpendicularis in CA, deſcribetur alia recta perpendicularis in CS; cujus Diſtantia a C æqualis erit Diſtantiæ prioris a puncto S.

Corol. IV. Curvæ omnes, Methodo *Prop. primæ* deſcriptæ, poſſunt delineari ſi loco anguli FCO ſubſtituatur recta circa Centrum C volubilis. Quippe ſi in figura *Prop. primæ* ſumatur angulus SCI = FCO & angulus QSU = ISC, punctum U concurſus crurum CO & SU deſcribet rectam BU: Unde motu anguli HSU circa S ut Polum, & rectæ CO circa C, deſcribetur eadem Curva quæ *Prop. prima* deſcribebatur; ſi nimirum ducatur CO & SL Concurſus U per rectam infinitam BU, Concurſus ejuſdem CO cum altero crure SH deſcribet Lineam, ſecundi Ordinis, *Propoſitionis primæ* Methodo deſcriptam.

PROP. IV.

Prop. IV.

Lineam Secundi Ordinis ducere per data quævis quinque puncta; & Lineam datæ ſpeciei per quatuor quævis puncta deſcribere.

IMprimis dentur quævis quinque puncta A, B, C, S, D, (*Fig.6.*) per quæ ducenda ſit Conica ſectio; ducantur rectæ CA, CS, SA, & circa Polos C & S rotentur Anguli dati ACS, ASC, & applicentur crurum CA & SA Concurſus primum ad punctum B, dein ad punctum D, & interea notentur puncta Q & R ubi crura CF & SK ſe mutuo decuſſant; per puncta Q & R ducatur recta infinita EQ, & ſi crurum CF, SK Concurſus ſemper ducatur per rectam QE, aliorum crurum concurſus deſcribet Lineam ſecundi Ordinis per data quinque puncta tranſeuntem; quippe ubi crurum CF, SK Concurſus accedit ad Q & R, ſectio conica tranſit per B & D; quando vero coincidunt crura CF & SK cum CS curva tranſit per A & neceſſario quoque tranſibit per C & S per *Corol.* I. *Prop.* I. curva igitur ducitur per data quinquepuncta.

Quod ſi dentur tantum quatuor puncta, infinitæ Lineæ ſecundi Ordinis per ea duci poſſunt; curva vero datæ ſpeciei, quæ per quatuor puncta duci poteſt, ſic determinatur.

Ducenda ſit imprimis Parabola per quatuor puncta C, S, L, T: (*Fig.7.*) Jungantur ut prius puncta C, S, L & circa data puncta C & S ut Polos rotentur anguli LCS & LSC; applicetur crurum CO & SH concurſus puncto quarto T, & interea concurrant reliqua crura bina CF & SK in A. Dein per C & S deſcribatur arcus Circuli cui inſcribi poſſit Angulus æqualis ſupplemento datorum LCS & LSC ad duos aut quatuor rectos; & ex determinato puncto A ducatur recta AE circulum hunc contingens. Si concurſus crurum CF & SK ducatur per hanc rectam, concurſus crurum CO & SH deſcribet Parabolam per dicta quatuor puncta tranſeuntem; ut ex Prop. ſecunda ſatis eſt manifeſtum.

Ducenda ſit nunc Hyperbola per data quatuor puncta cujus Aſſymptotωn angulus datur MIO (*Fig.8.*) ejuſque ſupplementum ad duos rectos MIU; cæteris manentibus ut in conſtructione præcedente ducatur ex A recta AQ circulo dicto occurrens in punctis N & Q & arcum NQ abſcindens in quo inſcribi poſſit angulus datus MIU; ſi crurum CF, SK

concurſus ducatur per hanc rectam, Hyperbola deſcribetur per data quatuor puncta tranſiens & datæ ſpeciei, motu puncti P concurſus crurum CO & SH.

Si vero ex data axium ratione Hyperbola vel Ellypſis ſit determinanda; id ex Corol. primo, Prop. ſecundæ non difficulter conficitur.

Quippe ſint ut prius *(Fig. 9.)* C, S, L, T, quatuor puncta data, & rotentur Anguli LCS & LSC circa Polos C & S; applicetur Concurſus crurum CL & SL ut prius puncto T, & interea notetur punctum A ubi concurrunt crura CF & SK; deſcribatur Circulus quoque per C & S cui idem angulus inſcribi poterit, datorum nim. LCS & LSC ſupplementum ad duos aut quatuor rectos, cujus centrum O dabitur & Radius OY in quo inveſtigari poteſt punctum X hac ratione. Ex data ſpecie Curvæ datur ratio Linearum GX & YX, quæ ſunt ut latera Quadrata axium per Corol. I. Prop. II. Proinde datur ratio OX ad OY, adeoque cum detur OY, dabitur OX. Centro O, radio vero OX, alius deſcribatur circulus, cui tangens ducatur ex jam determinato puncto A: Si Concurſus crurum CF & SK ducatur per hanc rectam Circulum tangentem, Concurſus crurum reliquorum CL & SL deſcribet Curvam datæ ſpeciei per data quatuor puncta tranſeuntem. Hæc eſt Conſtructio quam abſque Demonſtratione protulit D. *Neutonus* in Schol. *Prop.* 27. *Lib.* 1. *Princip.* ea vero ex *Corol.* 1. *Prop.* 2. facile conſequitur.

Corol. I. Cum ex puncto quovis A duæ duci poſſint tangentes Circulo; patet duas Conicas ſectiones, ejuſdem ſpeciei, per data quatuor puncta poſſe duci. Unica tantum ſectio Conica per quinque data puncta duci poteſt; nam ſi duæ per eadem quinque puncta tranſirent, duæ Lineæ ſecundi Ordinis ſe mutuo ſecare poſſent in pluribus punctis quam quatuor.

Corol. II. Eædem Curvæ deſcribi poſſunt variis admodum Angulis & diverſarum Rectarum Ope. Quippe eædem deſcriberentur Curvæ ſi junctis C, S & B vel D, A, C diverſi ſumerentur Anguli, & rectæ quoque inveſtigarentur diverſæ ab iis quas nos adhibuimus.

Curvæ *Prop.* I. deſcriptæ integrum curvarum ſyſtema conſtituunt; atque eæ ſolæ Linearum Ordinem ſecundum complent. Hæ fuerunt veterum Geometrarum deliciæ, quarum ſpeculationi plurimum admodum indulſerunt; nec ullo ſuperſedebant Labori, ut earum Proprietates licet maxime Complexas rimarent, & ſecundum Methodum ſuam demonſtrationibus elegantibus admodum ornarent. Harum vero Geometria tandem quantum fieri poteſt perfecta reddita eſt, ſublimiori horum ſeculorum Geometra, qui earum Theoriam abſolvit, variis iis viis quibus ſectiones conicas, vel datis umbilicis, vel iis non datis, determinavit in *Princip.* *Lib.* 1. Sectionibus 4 & 5.

Egregius

Egregius vero idem Auctor, non tantum curvarum primi generis Contemplationem ita abſolvit, ut parum admodum aliis ſuperſit indagandum; ſed primus aliud novum Curvarum Syſtema, Lineas ſcil. tertii Ordinis, enumerare aggreſſus eſt; harum Proprietates generales Sectionum Conicarum Proprietatibus conſimiles protulit, ipſaſque in claſſes atque ſpecies pulcherrime redegit, in eximio quem de Enumeratione Linearum tertii Ordinis conſcripſit Tractatu; earum numerus longe major, aut Figura magis complexa, longe difficilius determinanda, non deterruit acre Viri ingenium ad ſumma quævis feliciter aſſequenda nati. Curvis vero primi Generis jam deſcriptis, ad novas has delineandas progredimur.

SECTIO II.

De Deſcriptione Linearum Tertii Ordinis quæ punctum duplex habent.

PROP. V.

Moveatur ut in Prop. I. *Angulus datus* FCO (*Fig.* 10.) *circa Polum* C; *& interea Angulus datus* LNQ *ſemper percurrat, angulari ſuo puncto* N, *Rectam infinitam* AE, *ita ut Crus* NQ *perpetuo tranſeat per datum punctum* S; *ſi Concurſus crurum* CF *&* SN *ducatur per rectam infinitam* BQ *concurſus crurum* CO *&* NL *deſcribet Lineam tertii Ordinis.*

SIT punctum Q concurſus crurum CF & SN, punctum P concurſus crurum CO & NL; & ex punctis P, N, Q ducantur PM, NH, QK normales in datam CS; ſint quoque Anguli PRC, QUC æquales ſibi mutuo atque angulo dato FCO. Ducatur NT conſtituens angulum STN æqualem dato SNP, & ex S in NT cadat perpendicularis Sg, quæ

producta occurrat perpendiculo NH in I. Sit Pe parallela rectæ CS curvæ occurrens in P atque rectæ NH in e.

Dicatur $CS = a$, $BC = b$, $PM = y$, $CM = x$, $SA = e$; ſitque anguli FCO ſinus ad ejus coſinum ut d ad a; ſit $BK : KQ :: a : c$, $AH : HN :: a : f$, & ſinus anguli SNL ad coſinum ut g ad a.

His poſitis, eadem ratione quâ inveſtigavimus Valorem primum perpendicularis QN in Prop. I. deprehendemus $QK = \frac{bc \times \overline{dx + ay}}{a \times \overline{d + c} \times x + \overline{a^2 - dc} \times y}$ ubi ſigna aliqua mutanda ſunt ſi angulus FCO obtuſus ſit: Eſt vero $CK = BK - BC =$ (cum $BK : QK :: a : c$) $\frac{bdcy - bacx}{a \times \overline{d + c} \times x + \overline{a^2 - dc} \times y}$

& $SK = CS - CK = \frac{\overline{a^2 d + a^2 c + bac} \times x + \overline{a^3 - adc - bdc} \times y}{a \times \overline{d + c} \times x + \overline{a^2\, dc -} \times y}$.

Sed $SK : QK :: SH : HN :: SA - AH : HN$. Unde cum $AH : HN :: a : f$ erit $HN = \frac{efbc \times \overline{dx + ay}}{\overline{a^2 d + a^2 c + bac} \times fx + \overline{a^3 - adc - bdc} \times fy + abc \times \overline{dx + ay}}$

Sit $a^2 d + a^2 c + bac = m$ & $a^3 - adc - bdc = n$ eritque $HN = \frac{efbc \times \overline{dx + ay}}{mfx + nfy + abc \times \overline{dx + ay}}$ & $SH = \frac{emfx + nefy}{mfx + nfy + abc \times \overline{dx + ay}}$.

Cum Angulus externus STN æqualis ſit duobus TGN, TNG, & Angulus SNL (qui ſupponitur æqualis ipſi STN) ſit æqualis duobus TNS, TNG; patet $TGN = TNS$, adeoque Triangula SNg, NGH, NPe eſſe ſimilia; cumque Angulus $STg = NTH$ manifeſtum eſt Triangula STg, NTH, NIg ſimilia eſſe: Proinde erit Sg. Ng :: Ne : Pe. ST : NI :: Sg : Ng adeoque ST : NI :: Ne : Pe. Dicatur $HN = z$ eritque $ST = SH - HT = e - \frac{az}{f} - \frac{az}{g}$. $NI = NH + HI = z + \frac{ea}{g} - \frac{a^2 z}{gf}$. $Ne = z - y$ & $Pe = a + x - e + \frac{az}{f}$. Et proinde $e - \frac{az}{f} - \frac{az}{g} : \frac{ea}{g} + \frac{\overline{fg - a^2}}{fg} z :: z - y : a + x - e + \frac{az}{f}$. Loco z ſubſtituatur valor prius repertus rectæ HN & prodibit $efg \times \overline{mfx + nfy + abc \times \overline{dx + ay}} - aefbc \times \overline{g + f} \times \overline{dx + ay} : eaf \times \overline{mfx + nfy + abc \times \overline{dx + ay}} + \overline{fg - a^2} \times efbc \times \overline{dx + ay} :: efbc \times \overline{dx + ay} - mfxy - nfy^2 - abc \times \overline{dxy + ay^2} : \overline{a - e} \times \overline{mfx + nfy + abc \times \overline{dx + ay}} \times \overline{a - e} + aebc \times \overline{dx + ay} + mfx^2 + nfyx + abc \times \overline{dx^2 + ayx}$.

Unde ducendo media & extrema in ſe mutuo prodibit æquatio hujus formæ $Ay^3 + Bxy^2 + Cx^2y + Dx^3 + Ey^2 + Fxy + Gx^2 = o$, cujus coefficientes A, B, C, D, E, F, G calculo obvio ex Analogia inventa inveſtigari poſſunt.

Cum vero variabiles Quantitates x & y in hac æquatione ad tres tantum dimenſiones aſcendant, manifeſtum eſt Lineam deſcribi Ordinis tertii quæ in tribus punctis Rectâ ſecari poteſt.

Corol. I. Curva habet punctum duplex in C; Quippe ſi in æquatione curvæ evaneſcat y, manebunt tantum termini Dx^3 & Gx^2, & æquatio erit formæ $Dx^3 + Gx^2 = o$; cujus radices tres ſunt o, o, $-\frac{G}{D}$; unde cum C ſit principium abſciſſæ, curva tranſibit neceſſario per C bis. Curva vero nunquam tranſit per S, cum curvæ puncta ſemper verſentur in recta NL, quæ nunquam poteſt tranſire per S; alias duæ rectæ Lineæ SN, NL ſe mutuo ſecare poſſent in duobus punctis N & S.

Corol. II. Super recta Linea *(Fig.* 11.*)* CS deſcribatur ſegmentum Circuli cui inſcribi poſſit angulus datus SNL; ſecet Circulus hic rectam AE in duobus punctis N & *n*, & jungantur CN, C*n* atque SN, S*n* occurrentes rectæ BQ in punctis Q, *q*. Ducantur CQ, C*q*, atque rectæ CP, Cπ angulos datos PCQ, πC*q* æquales ipſi FCO conſtituentes; tunc rectæ CP & Cπ tangent arcus curvæ concurrentes in puncto duplice C, & ſe mutuo decuſſantes ad formam *Nodi* vel *Crucis*.

Corol. III. At ſi Recta AE Circulum in quo inſcriptus eſt angulus datus SNL contingat in puncto quovis D, evaneſcet Ovalis quæ jacebat inter CP & Cπ, & curva evadet *Cuſpidata* in C.

Corol. IV. Quod ſi recta AE cadat penitus extra circulum nec eum contingat nec ſecet; curva habebit *Punctum* conjugatum in C abſque nodo, cruce, cuſpide vel ovali.

Corol. V. Quod ſi, cæteris manentibus ut in Propoſitione, *(Fig.* 12.*)* Concurſus cruris CF cum alia SZ angulum datum cum SN conſtituente, ducatur per rectam infinitam XZ, punctum P deſcribet adhuc Lineam tertii Ordinis punctum duplex habentem in C: Nam hæc conſtructio ad eam hujus Propoſitionis reduci poteſt; Quippe ſi ſumatur CSD æqualis NSZ, & circa C moveatur datus angulus QCZ æqualis ipſi SCD, & crurum CZ, SZ concurſus ducatur per rectam XZ, concurſus crurum CQ & SN deſcribet aliam rectam per *Prop.* 3. Unde hæc conſtructio ad eam hujus Propoſitionis facile reduci poterit.

PROP.

PROP. VI. PROBLEMA.

Determinare Curvarum Aſſymptotos & crura infinita.

SUper Rectam (*Fig.* 13. *n.* 1.) CS deſcribatur ſegmentum Circuli cui inſcribi poſſit Angulorum datorum FCO, LNS ſupplementum ad quatuor rectos. Supponamus imprimis rectam BQ huic Circulo occurrere in duobus punctis Q & I: Cumque angulus SQC ſit ſupplementum angulorum FCO, SNL ad quatuor rectos, manifeſtum eſt crura CP & NL evadere parallela ubi concurſus crurum CF & SN in Q circulum tangit, & proinde tunc curvam abire in infinitum. Id vero bis contingit, cum recta circulo bis occurrat; adeoque ſi anguli QCP, ICπ æquales ſint dato FCO, erunt CP & Cπ parallelæ duabus curvæ Aſſymptotis; quarum Poſitio determinatur eadem fere ratione qua Aſſymptotos Linearum ſecundi Ordinis in *Prop.* II. determinavimus.

Sit G punctum ubi Aſſymptotos occurrit rectæ CS; & GC erit ad GH ut QC*q* ad QS*q* ſi puncta Q, *q* ſint quamproxima; GC vero erit ad CH ut QC*q* ad QC*q* — QS*q* ſive ut BC × SQ² ad BC × SQ² — BS × CQ². Unde cum detur angulus PCS quem Aſſymptotos per G ducta conſtituit cum CG, dabitur Aſſymptotos Poſitione; & ſimiliter determinatur Aſſymptotos quæ parallela eſt rectæ Cπ.

Adhæc curva abit in infinitum cum N ſit punctum infinite diſtans; nam P verſatur ſemper in Recta NL: Proinde per S (*n.* 2.) ducatur SU parallela AN, occurrens Rectæ BQ in R; ducatur CR, & dein Recta CZ angulum ZCR conſtituens æqualem dato FCO, eritque CZ parallela tertiæ Aſſymptoto Curvæ. Unde Curva habebit ſex crura infinita Hyperbolica ad tres Aſſymptotos.

Sed 2°. Si recta BQ (*n.* 3.) circulum contingat in Q evadet QC*q* = QS*q* ex natura circuli, adeoque GC quæ eſt ad CH ut QC*q* ad QC*q* — QS*q* evadet infinita; Aſſymptotos proinde abit in infinitum & nullibi reperiri poteſt. Quod ſi AN non ſit parallela rectæ SQ curva habebit duo crura Hyperbolica ad Aſſymptoton rectæ CZ parallelam, & duo Parabolica quæ abibunt in plagam rectæ CP, ſi QCP = FCO.

3°. Si AN ſit parallela rectæ SQ vel SI, (*Fig.* 14.) evadet CH infinita, adeoque cum GC : CH :: BC × SQ² : BC × SQ² — BS × CQ², erit CG quoque infinita, & proinde crura evadunt Parabolica abeunte Aſſymptoto in infinitum. Cumque in hoc caſu abeat N in infinitum quando Q circulum

circulum tangit; patet curvam quatuor habituram crura infinita duo Parabolica & totidem Hyperbolica.

In duobus igitur caſibus curva fit Hyperbolo-Parabolica; cum ſcil. BQ circulum contingit nec tamen AN parallela fit SQ, atque cum BQ circulum ſecat & AN exiſtit parallela alteri rectarum SQ, SI.

Quod ſi 4°. BQ circulum contingat & ſimul AN parallela ſit ipſi SQ curva duo habebit crura infinita Parabolica eritque *Neutonianarum* ſpeciei 68, 69 vel 70.

Denique cadat BQ tota extra circulum, & duo deſcribentur curvæ crura infinita abeunte puncto N in infinitum; ea vero Hyperbolica erunt, & curva erit Hyperbola defectiva, claſſis quintæ tertii Ordinis, ſecundum Enumerationem D. *Neutoni.*

Corol. I. Si angulus FCO (*Fig.* 15.) ſupplementum ſit anguli SNL ad duos rectos, angulus CQS evaneſcet vel evadet æqualis duobus rectis; atque crura NP, CP evadunt parallela cum Q coincidit cum B & cum Q abit in infinitum. Aſſympototos cruris quod formatur in primo caſu ſic determinatur: Coincidat CO cum CS, & ſumatur punctum O in recta CS ita poſitum ut OA ſit ad CA, ut eſt CB ad CS; & per O duc rectam O*p* parallelam cruri CF erit O*p* curvæ Aſſymptotos. Quippe ſupponamus Q & N eſſe punctis B & A quam proxima, adeoque *p* punctum curvæ concurſui ejus cum Aſſymptoto ſua quamproximum, atque ſi *p*O ducatur parallela rectæ CF ea erit curvæ Aſſymptotos. Sed angulus O*p*C = FC*o* = BCQ & O*p*N = AL*n* = ASN, cum triangula A*n*L & S*n*N æquiangula ſint. Sed anguli O*p*C, O*p*N (qui evaneſcunt cum *p* abit in infinitum) erunt ut rectæ OC & OA; anguli vero evaneſcentes (ipſis O*p*C, O*p*N æquales) BCQ, ASN ſunt ut rectæ SB & CB; & proinde eſt OC ad OA, ut SB ad CB; adeoque OA ad CA, ut CB ad CS.

Altera vero Aſſymptotos, (*Fig.* 16.) ad quam crura duo deſcribuntur quando Q abit in infinitum, ſic determinatur poſitione. Ducantur CF & SN parallelæ rectæ BQ, & ſint anguli SND, FCG æquales datis FCO, SNL; & ſumatur CU ad CD, ut CB ad CS, eritque Recta *p*U ducta per U parallela ipſi CG altera Curvæ Aſſymptotos; cujus rei demonſtrationi, cum ſimilis ſit præcedenti, non opus eſt immorari.

Corol. II. Cæteris manentibus ut in *Corol.* præcedente, Si BQ parallela ſit ipſi CS, Aſſymptoti O*p*, U*p* abeunt ad diſtantiam infinitam, & crura deveniunt Parabolica; nam in eo caſu CB evadit infinita; ſed OA : CA :: CB : CS & CU : CD :: CB : CS; adeoque OA & CU evadunt infinitæ; proinde duo Aſſymptoti in infinitum abeunt & earum crura evadunt Parabolica. Alia vero deſcribuntur duo crura Hyperbolica cum punctum N abit in infinitum; adeoque curva evadit Hyperbolo-Parabolica ſpeciei, ſecundum Enumerationem *Neutonianam* 47, 48,

49, 51

49, 51 vel 54; cum hæ ſolæ habeant punctum duplex ad finitam diſtantiam, & ſimul duo crura Parabolica atque duo Hyperbolica.

Corol. III. Si AN & BQ ſint parallelæ erit *(Fig.* 16.*)* CD infinita, adeoque & CU; proinde Aſſymptotos curvæ, ad quam crura deſcribuntur cum Q abit in infinitum, nullibi reperitur, & crura ea curvæ evadunt Parabolica; cumque N abeat in infinitum una cum Q manifeſtum eſt curvam dicta habere crura infinita Parabolica & duo tantum præterea Hyperbolica ad Aſſymptoton O*p*. Quod ſi ſint AN & BQ parallelæ quoque ipſi CS, curva habebit duo tantum crura pure Parabolica; eritque ſpeciei 68, 69, vel 70.

Corol. IV. Quod ſi maneat BQ ut in *Corol.* I, & fiat AN parallela ipſi CS, erit CA infinita; ſed OC : CA :: BS : CS; & proinde OC quoque infinita erit; adeoque ea crura quorum duo erant ad Aſſymptoton O*p*, & duo deſcribebantur cum N abibat in infinitum, nunc coincidunt (quoniam cum N infinite diſtat Q fit in B) & evadunt Parabolica. Manet vero altera Curvæ Aſſymptotos Recta nim. *p*U, adeoque Curva eſt Hyperbolo-Parabolica ut in prioribus *Corolariis*.

Corol. V. Si anguli FCO, SNL *(Fig.* 13.*)* ſint recti, altera Aſſymptoton erit perpendicularis ipſi CS, altera perpendicularis rectæ BQ.

Lemma I.

Si datus Angulus SNP (Fig. 17.) *angulari ſuo puncto* N *percurrat Rectam infinitam* AE, *&* *ex dato puncto* S *ducatur perpetuo* SR *angulum datum* NSR *cum recta* SN *conſtituens, ea ſemper occurret rectæ* NP *productæ in puncto aliquo aliæ Rectæ* RF *Poſitione datæ.*

Sit R punctum ubi recta SR occurrit rectæ NP & ſi angulus SFN = dato SRN circulus per N, S, R ductus tranſibit per F; adeoque angulus RFN = RSN, cum vero detur punctum F & angulus RFN (=dato RSN) dabitur Recta FR, quam perpetuo tangit punctum R concurſus Rectarum NP & SR.

Corol. Hinc viciſſim ſi duæ rectæ datæ AE, FR ſe mutuo ſecent in puncto F, & ex S educatur SR datæ FR occurrens in puncto R, atque ad id punctum conſtituatur RP, ita ut angulus SRP = SFE, occurrat vero RP datæ AE in N, angulus SNP dabitur eritque ſemper æqualis angulo SFR.

PROP.

Prop. VII.

Curvæ omnes Prop. V. *deſcriptæ poſſunt facilius duci ſi cæteris manentibus, loco anguli dati* FCO *revolvatur recta Linea* CO *circa Polum* C, *cujus Concurſus cum* NP *Lineam tertii Ordinis deſcribet, ſi ejuſdem Concurſus cum* SN *ducatur per Rectam datam.*

CÆteris manentibus ut in *Prop.* 5. ſumatur angulus KCS (*Fig.* 18.) æqualis angulo FCO & ducatur SK; Ex S exeat recta S*q*, angulum RSN conſtituens æqualem CSK, occurrens rectæ NP productæ in R & radio CO in *q*. Manifeſtum eſt ex *Lemmate* 1. R ſemper fore in recta data ER, atque angulum SRP dari: Patet quoque ex *Prop.* 3. crura CO & S*q* concurſu ſuo *q* rectam percurſura cum CQ & SQ concurſu ſuo deſcribant rectam BQ, & ſimul coincidant CO & S*q* cum CS quando Q pervenit ad K. Proinde punctum Curvæ P determinatur concurſu Radii CO, circa C ut Polum revolventis, cum RP crure dati anguli SRP angulari ſuo puncto R rectam FR percurrentis, dum ejuſdem radii CO concurſus cum crure SR (punctum nimirum *q*) percurrit aliam rectam G*q*. Hinc igitur ſimplicior evadit deſcriptio, dum loco FCO circa C revolventis Radius rectilineus CO ſubſtituitur.

Corol. I. Simili plane ratiocinio demonſtrari poſſet omnes curvas *Prop.* 5. deſcribi poſſe licet alteruter angulorum FCO, SNP ſupponatur ſemper rectus aut datæ cujuſvis Magnitudinis.

Corol. II. Hujus *Prop.* ope failius ſpecies curvarum deteguntur. Super CS (*Fig.* 19.) deſcribatur circulus habens angulum datum SNP inſcriptum; huic circulo occurrat BQ in punctis duobus Q & L, atque rectæ CQ, CL parallelæ erunt curvarum Aſſymptotis duabus, & denotant plagas quatuor crurum Hyperbolicorum. Sumatur QH in recta SQ ita ut QH : NQ : : QC*q* : QS*q*—QC*q*, & ducatur per H recta infinita parallela ipſi CQ atque ea erit Aſſymptotos curvæ, & ſimiliter altera Aſſymptotos quæ parallela eſt rectæ CL determinatur. Porro (*Fig.* 20.) ducatur SG parallela rectæ AE occurrens rectæ BQ in G; dein ducatur CG atque ei parallela erit tertia curvæ Aſſymptotos.

Corol. III. Ex his atque *Corol.* 2, 3 & 4, *Prop.* 5. ſpecies curvarum inveſtigari poſſunt: Quippe ex *Corol.* præcedente curvarum crura inoteſcunt, & ex *Corol. Prop.* 5. conſtare poteſt an curva ſit *Nodata*, *Cuſpidata* an *Punctata*. Curva nimirum habebit *Nodum* vel *Crucem* quoties AN occurrit bis circulo priore *Corollario* deſcripto; *Cuſpidem* vero quoties AN dictum circulum contingit, & *Punctum* conjugatum ſi tota extra circulum ponatur.

Corol. IV. Hinc 1. Si rectæ BQ & AN circulum ſecent, nec ſit AN parallela alterutri rectarum SQ vel SL, Curva habebit tres Aſſymptotos, ſex crura Hyperbola, & duos arcus ſe mutuo ſecantes in C vel ad formam *Nodi* vel ad formam *Crucis*, atque curva deſcripta erit aliqua *ſpecierum* 2, 7, 8, 25, ſi diametro careat, ſed *ſpecierum* 11, 18, 19, 20, ſi diametrum habeat.

2. Si BQ Circulum ſecet, AN vero eum contingat, Curva habebit ſex crura Hyperbolica ad tres Aſſymptotos ut prius, ſed *Cuſpidata* erit in C, atque *ſpeciei* 3 vel 12; hujus ſcil. cum diametrum habet, illius vero cum diametro caret.

3. Quod ſi BQ circulum ſecet, AN vero integra extra circulum ponatur, curva adhuc conſtabit ex tribus Hyperbolis, & *Punctum* conjugatum habebit in C abſque nodo, cruce, Cuſpide vel Ovali, & *ſpeciei* erit vel 4, vel 13.

4. Nunc tangat recta BQ circulum eumque ſecet AN; nec ſit AN parallela rectæ SQ aut SL; tunc Curva habebit quatuor crura infinita duo Parabolica & duo Hyperbolica cum *nodo* vel *cruce* & *ſpeciei* erit 47, 51, ſeu 54.

5. Quod ſi BQ adhuc circulum contingat & AN ei quoque tangens ſit; *Nodus* in *Cuſpidem* mutabitur & figura Hyperbolo-Parabolica erit *ſpeciei* 48.

6. Contingat adhuc BQ circulum, cadat vero AN tota extra Circulum, & figura Hyperbolo-Parabolica habebit *Punctum* conjugatum in C abſque nodo, Cuſpide vel Ovali eritque curva *ſpeciei* 49.

7. Cadat nunc recta BQ penitus extra Circulum, at eum ſecet AN; atque curva habebit duo tantum crura Hyperbolica ad unicam Aſſymptoton parallelam rectæ CG & *Nodum* in C cum Ovali eritque *ſpeciei* vel 34 vel 41; hujus ſcil. cum Diametrum habet, illius vero ſi Diametro careat.

8. Contingat AN circulum ſed BQ extra eum ponatur, & *Nodus* mutabitur in *Cuſpidem* acutiſſimum, atque curva erit Hyperbola defectiva *ſpeciei* vel 35 vel 42; in hoc caſu curva ſit Ciſſois veterum & Diametrum habet in illo Diametro caret.

9. Cadat AN extra Circulum ſimul ac BQ & *Nodus* cum Ovali abibit in *Punctum* conjugatum & Hyperbola erit defectiva *ſpeciei* vel 36, 43 vel 44.

10. Sit AN parallela rectæ SQ vel SL, & NQ evadet infinita, adeoque & HQ; unde ſi BQ ſecet circulum curva habebit duo crura infinita Hyperbolica & duo Parabolica; atque Lineæ deſcribi poſſunt ſecundum varias Poſitiones rectæ AN *ſpecierum* 47, 51, 48, 49: Quod ſi BQ circulum contingat curva habebit duo tantum crura infinita Parabolica; eritque Parabola *Nodata* ſi AN circulum ſecet, adeoque *ſpeciei* 68, *Cuſpidata* ſi AN circulum contingat, adeoque *ſpeciei* 70, & denique *Punctata* ſi AN neque circulum ſecet nec eum contingat, & proinde *ſpeciei* 69.

11. Abeat punctum C in infinitum, & moveatur Recta CO (*Fig.* 21.) ſibi ſemper parallela, ejuſque concurſus cum SN ducatur perpetuo per Rectam datam BQ, & concurſus ejuſdem cum NL altero Crure dati anguli SNL deſcribet curvam quæ punctum duplex habet ad diſtantiam infinitam, & D. *Neutono* Hyperboliſmus Hyperbolæ dicitur; cujus tres Aſſymptoti ex deſcriptione facile determinantur. Ducatur 1. per datum punctum S Recta SB parallela datæ AN, & per punctum B ducatur EB parallela rectæ CO, atque recta EB erit Aſſymptotos Curvæ; nam cum Q accedit ad B, punctum N atque Recta NL abeunt in infinitum, adeoque etiam punctum Curvæ quod in recta NL ſemper verſatur; unde EB evadit tangens Curvæ ad Diſtantiam infinitam. 2. Ex dato puncto S ducatur recta SK conſtituens angulum SKB æqualem Differentiæ datorum SNL, OQB; & ſi ad punctum K conſtituatur angulus SKR æqualis dato SNL erit KR alia Curvæ Aſſymptotos; quippe accedat Q ad K, & coincidet CO cum KR, adeoque ob angulos SKR, SNL æquales, in hoc caſu erit CO parallela rectæ NL; proinde punctum curvæ P, quod determinatur concurſu rectarum CO & NL abibit in infinitum, & recta KR curvam tanget ad diſtantiam infinitam. 3. Ducatur SF parallela datæ BQ occurrens rectæ AN in F, ſit angulus SFI=SNL & ſumatur in recta SF Linea FU ad FI, ut eſt IG ad IK; ſi per punctum U ducatur UL parallela recta FI, ea erit tertia curvæ Aſſymptotos. Curvarum ſpecies quæ hac ratione deſcribi poſſunt ſunt 57, 58, 59, 60; quarum Figuras delineatas in ſæpe citato D. *Neutoni* Tractatu cernere licet.

12. Cæteris manentibus ut in præcedenti caſu ſit angulus SBT=SNL —OQB, & coincident puncta K & B, adeoque etiam duæ Aſſymptoti coincident quæ per ea puncta tranſibant; Curva vero evadit ea quæ D. *Neutono* Hyperboliſmus Parabolæ appellatur, quæ quatuor habet crura infinita Hyperbolica ad Aſſymptotos EB & UL; eſtque *ſpeciei* 64, ſi diametro careat, ſed *ſpeciei* 65, ſi diametrum habeat.

13. Duo denique ſunt caſus in quibus Hyperboliſmus migrat in Parabolam *Carteſianam*, vel ſcil. ſi AN & BQ ſint parallelæ, vel ſi angulus SNL=OQB: In priori caſu coeunt puncta B & F & in infinitum una abeunt, unde crura hyperbolica quæ erant ad Aſſymptotos EB, UL in

duo migrant Parabolica; in poſteriori coincidunt Aſſymptoti KR, UL, & una in infinitum abeunt, eorumque crura evadunt parabolica; in utroque curva eſt *ſpeciei 66*.

Corol. V. Curva ſemper tranſit per punctum A; nam cum N pervenit ad A tum Q ſemper accedit ad B & CO ad poſitionem CA, unde CO occurret NP in A, earum vero Occurſu Curvæ puncta determinantur.

PROP. VIII. PROBLEMA.

Æquationes Curvarum ad aliquam quatuor Neutonianarum *reducere atque inde ſpeciem Curvæ detegere.*

UT Æquatio curvæ poſſit obtineri quæ ad aliquam quatuor D. *Neutoni* facile reduci poterit ea indaganda eſt ad Ordinatas alicui Aſſymptotωn parallelas; quod ad Ordinatas tertiæ Aſſymptoto, *Corol.* 2. Prop. præcedentis determinatæ, parallelas ſequenti fit calculo.

Sit SG *(Fig. 22.)* parallela rectæ AN occurrens datæ rectæ BQ in G, & ducatur CG; ea erit Aſſymptoto curvæ parallela. Ducatur CB angulum ACB conſtituens æqualem dato SNP; huic parallela ſit SD, & rectæ CG parallelæ ducantur PM, QT, N*e*; dicatur CB $= b$, CG $= d$, SD $= a$, CD $= e$, eritque DG $= d - c$; ſit AG $= e$, PM $= y$, CM $= x$. Quoniam PM : CM :: QT : CT & BT : QT :: BC : CG :: $b : d$ erit QT $= \frac{bdy}{dx + by}$ & RQ $= \frac{cdx + \overline{c - d} \times by}{dx + by}$ & CT $= \frac{bdx}{dx + by}$ atque SR (= CT + SD) $= \frac{\overline{a + b} \times dx + aby}{dx + by}$. Sed SR : RQ :: SU : NU, atque ſi ducatur N*o* parallela rectæ SD erit (ob ſimilia Triangula AN*o*, SDG) A*o* : N*o* (= UD) :: GD : SD :: $d - c : a$ :: AG — N*u* : S*u*; unde calculo obvio prodibit NU $= \frac{ae \times \overline{cdx + \overline{c - d} \times by}}{\overline{d^2a + d^2b - dbc} \times x}$.

Ducatur NK conſtituens angulum NKU = NUK = SNP, & dabitur angulus KNU, adeoque ratio NU ad UK; quæ ſit ut a ad f eritque UK $= \frac{f}{a}$ NU, & SK $= \frac{ae \times \overline{a + b - fec} \times dx + \overline{a^2be - feb} \times \overline{c - d} \times y}{\overline{d^2a + d^2b - dcb} \times x}$

Sed

Sed angulus NSK $=$ P*N*e quoniam angulus PN*e* cum SN*u* ſupplementum conſtituit anguli SNP ad duos rectos, atque NSK cum SNU ejuſdem anguli idem ſupplementum conficiat; angulus etiam P*e*N $=$ SKN, & proinde ſimilia ſunt triangula KNS, *e*NP; atque SK : KN ($=$ NU) :: N*e* ($=$ CD $-$ NU $-$ PM) : P*e* $=$ DU $-$ CM. Unde ſi $\overline{d^2a + d^2b - dbc} = m$ prodibit $ae \times \overline{a+b} - fec \times dx + \overline{a^2be - feb} \times \overline{c-d} \times y : ae \times cdx + b \times \overline{c-d} \times y :: cmx - mxy - ae \times cdx + b \times \overline{c-d} \times y : ae \times \overline{a+b} \times d - am \times x + a^2bey - mx^2$; unde ducendo media & extrema in ſe mutuo & terminos in ordinem redigendo prodit æquatio hæc

$$a \times \overline{a+b-fc} \times dmx^3 + \overline{a^2b - acd + fb \times \overline{c-d}} \times mx^2y + mab \times \overline{d-c} \times xy^2$$

$$\left.\begin{array}{l} + \overline{cm - aecd} \times acd \\ + ad \times \overline{a+b} + fcd \times am \\ - a^2de^2 \times \overline{a+b}^2 - afced^2 \times \overline{a+b} \end{array}\right\} x^2 \quad \left.\begin{array}{l} + \overline{cm - 2aecd} \times ab \times \overline{c-d} \\ - a^3dbe \times \overline{a+b} + fcda^2be \\ - aed \times \overline{a+b} - 2m \times \overline{a^2b - fb \times \overline{c-d}} \end{array}\right\} xy$$

$$- b^2a^2e \times \overline{c-d}^2 - a^4b^2e + a^2b^2ef \times \overline{c-d} \times y^2 = 0.$$

In hac Æquatione deeſt Terminus y^3 quoniam Ordinatæ parallelæ ſunt curvæ Aſſymptoto; ſed ut æquatio hæc ad formam *Neutonianam* reducatur reſtat adhuc ut termini x^2y, xy, y^2 exterminentur.

Caſus I. Ducatur BH parallela rectæ CG, ſumatur BH ad BC, ut $a^2b - cad + fb \times \overline{c-d}$ ad $2ab \times \overline{d-c}$; ducatur recta CH, ad quam producatur PM, donec ei occurrat in N, atque in Æquatione ad axem CH (ſi nunc CN $= x$ & PN $= y$) deprehendemus duos tantum reperiri terminos trium dimenſionum x^3 & xy^2 ſed x^2y evaneſcere.

Sit $a^2b - acd + fb \times \overline{c-d} = i$ & $2ab \times \overline{d-c} = k$; ſintque tres rectæ BH, BC, CH tribus i, k, g proportionales eritque CM ($= x$) $= \frac{kx}{g}$ atque PM (y) $=$ PN $-$ NM $= y - \frac{ix}{g}$. Proinde in æquatione generali modo inveſtigata loco x & y ſubſtituantur illi valores & prodibit æquatio in qua deerit terminus x^2y. Æquatio prius reperta eſt hujus formæ ſi b ſcribatur pro $ad \times \overline{a+b} - fcd$, &c.

$mbx^3 + mix^2y + \frac{mk}{2}xy^2 + qx^2 + uxy + ny^2 = 0$ pro x ſcribatur

$\frac{kx}{g}$.

$\frac{kx}{g}$ & pro y ejus valor $y - \frac{ix}{g}$ & prodibit $\frac{\overline{mhk^3 - \frac{mi^2k^2}{2}}}{g^3} \times x^3$

$+ \frac{mk}{2g} xy^2 + \frac{\overline{qk^2 - uki + ni^2}}{g^2} x^2 + \frac{\overline{uk - 2ni}}{g} xy + ny^2 = 0.$

Dein pro x ſcribas $x - \frac{2ng}{mk}$ & pro y ſcribe $y - \frac{uk + 2ni}{mk}$ atque termini xy & y^2 evaneſcent, & æquatio prodibit pura *Neutoniana* hujuſmodi.

$$xy^2 + \frac{2ng}{m^2k^2}\,\overline{2ni - uk} \times y = \frac{i^2k - 2bk^2}{g^2} x^3 - \left.\begin{matrix} \frac{6ni^2k - 2hk^2}{mkg} \\ + \frac{2uki - 2qk^2 - 2ni^2}{mkg} \end{matrix}\right\} x^2$$

Hæc æquatio ſolis Coefficientibus ab ea differt D. *Neutoni* $xy^2 + ey = ax^3 + bx^2 + cx + d$.

Caſ. II. Quod ſi in æquatione imprimis inventa ſint Termini x^3, x^2y at deſit xy^2, alia methodo opus eſt ut Æquatio ad *Neutonianas* reduci poſſit. Sit æquatio inveſtigata $mhx^3 + mix^2y + qx^2 + uxy + ny^2 = 0$, ſume BH : BC :: h : i & æquatio prodibit hujus formæ $\frac{hi^3}{g^3} - \frac{hi^3}{g^3} x^3$ $+ \frac{i^3}{g^2} x^2y$ &c. ubi terminus x^3 evaneſcit & manet tantum terminus x^2y trium dimenſionum; unde ad formam *Neutonianam* reducta æquatio erit hujuſmodi $xy^2 + ey = bx^2 + cx + d$, atque hæc eſt forma æquationum quoties BQ tranſit per D, nam in eo caſu $d = c$, & $k = 2ab \times \overline{d - c} = 0$; adeoque terminus $\frac{mk}{2} xy^2$ evaneſcit. Cum vero coincidant G & D, erit AH parallela rectæ SD, adeoque per *Corol.* 4. *Prop.* 7. *n.* 10. curva erit Hyperbolo-Parabolica; quæ probe conſentiunt cum §. 21 & 22. Enumerationis *Neutoni*, ubi oſtendit curvas eas eſſe Hyperbolo-Parabolicas quæ hac æquatione $xy^2 + ey = bx^2 + cx + d$ deſignantur.

Caſ. III: Quod ſi Terminorum x^2y & xy^2 uterque deſit, & æquatio ſit hujus formæ $hx^3 + qx^2 + uxy + ny^2 = 0$; ſumatur BH ad BC, ut u ad $2n$, & terminus xy exterminabitur; atq; æquatio induit hanc formam, $\frac{8hn^3}{g^3} x^3 + \frac{4qn^2 - u^2n}{g^2} x^2 + ny^2 = 0$, quæ eſt æquatio formæ, Caſus tertii apud *Neutonum*, atque Curvarum eſt Parabolicarum *ſpecierum* 68, 69 vel 70.

Caſ. IV.

Caſ. IV. Quod ſi Punctum duplex C abeat in infinitum, & Co moveatur ſibi ſemper parallela (*Fig.* 21.) Æquatio ad Ordinatas parallelas Rectæ EB (quæ per *Corol.* 4. *Prop.* 7. eſt Curvæ Aſſymptotos) erit hujus formæ.

$$Ax^3 + Bx^2 + CX + D = x^2y + Exy + F$$

ubi deſunt Termini xy^2, y^3, y^2, & proinde ſi abſciſſa ſumatur in Aſſymptoto UL, evaneſcet etiam Terminus Ax^3, & æquatio Curvæ reduci poterit ad formam $xy^2 + ey = cx + d$: Quod ſi coincidant K & B æquatio ad formam D. *Neutoni* reducta erit $xy^2 + ey = d$.

Caſ. V. Denique ſi Punctum duplex ſit ad Diſtantiam infinitam, ut in præcedenti Caſu, coincidant vero rectæ SK & SF, vel rectæ SB & SF, æquatio Curvæ erit formæ

$$Ax^3 + Bx^2 + Cx + D = xy + Fy.$$

quæ ad Æquationem Caſus 1. D. *Neutoni* loco x ſcribendo $x - F$ facile reducitur.

Schol. Quoniam Contemplationes cujuſcunque generis ſi maneant nimium generales, tandem tædio fiunt, ne hoc vitio laboret noſtra Curvarum Conſideratio, particulares aliquas Conſtructiones, ex generali *Prop.* 5. deductas, ſimpliciorum tertii Ordinis Linearum ſubjungere lubet; quarum demonſtrationes cuivis in Locorum Conſtructione vel leviter verſato ſatis obviæ erunt.

Ex. 1. Sit angulus SNP (*Fig.* 23.) rectus, & ſint Lineæ AN, BQ perpendiculares in C; ponantur puncta A & B inter C & S; ſitque CB $= b$, SA $= c$, CS $= a$, PM $= y$, CM $= x$, eritque

$$xy^2 + \frac{cby^2}{a-b} = \frac{\overline{a-b}}{b}x^3 + \frac{\overline{a-b}\times\overline{a-c}}{b}x^2.$$

Unde ſi ſit CD ad CB, ut SA ad SB, & DM dicatur x, prodibit æquatio formæ *Neutonianæ*,

$$xy^2 = \frac{\overline{a-b}}{b}x^3 - \overline{3c + \frac{\overline{a-c}\times\overline{a-b}}{b}}\,x^2 + \overline{\frac{3bc^2}{a-b} - 2c\times\overline{a-c}}\times x - \frac{b^2c^3}{\overline{a-b}^2}$$

$$+ \frac{bc^2\times\overline{a-c}}{a-b}.$$

Cujus radices ſi $y = 0$ erunt $\frac{cb}{a-b}$, $\frac{cb}{a-b}$, $\frac{cb}{a-b} + c - a$

$= \frac{ca}{a-b} - a$. Proinde cum radices majores ſint æquales quoties SA eſt minor SC, & tertia ejuſdem ſigni quoties SB eſt minor SA, manifeſtum eſt in hoc caſu curvam deſcribi *ſpeciei undecimæ*, atque tres habere Aſſymptotos, ſex crura Hyperbolica, diametrum CD atque *Nodum* in C. Aſſymptoton una eſt perpendicularis in CS ad punctum D, reliquæ duæ ſunt parallelæ rectæ CQ quando angulus CQS evadit rectus, earumque Occurſus cum CS facile determinari poteſt ex *Corol.* 2. *Prop.* 7. *v. Fig.* Curvæ apud *Neutonum*.

II. Coincidat

II. Coincidat C cum A, *i. e.* tranſeat recta $\overline{AN}$ per C; eritque æquatio Curvæ hujuſmodi quoniam $c = a$. $xy^2 = \frac{a-b}{b}x^3 - 3ax^2 + \frac{3ba^2}{a-b}x - \frac{b^2 a^3}{\overline{a-b}^2}$ cujus radices ſunt $\frac{ab}{a-b}$, $\frac{ab}{a-b}$, $\frac{ab}{a-b}$. Unde cum radices ſint omnes æquales, Hyperbola ſit *Cuſpidata* in C, atque *ſpeciei duodecimæ*.

III. Sit C inter S & A, ſeu ſit SA major SC, & æquatio eadem erit ac in primo exemplo & radices eædem; cum vero nunc c ſit major quam a manifeſtum eſt $\frac{cb}{a-b} + c - a$ majorem quam $\frac{cb}{a-b}$, adeoque radices duas minores nunc eſſe æquales, & proinde curvam eſſe *ſpeciei decimæ tertiæ* atque *Punctatam* eſſe in C. Eadem erit ſpecies Curvæ deſcriptæ cum AN verſus contrarias partes abit, & S intercedit punctis C & A.

IV. Sint rectæ AN & BQ inter C & S, ſitque SA minor quam SB, atque CD minor erit quam CB; & quoniam $a - b$ ($=$ SB) major eſt quam SA ($= c$) erit a major quam $\frac{ca}{a-b}$, adeoque radix $\frac{ca}{a-b} - a$ negativa erit. Unde cum duæ $\frac{cb}{a-b}$, $\frac{cb}{a-b}$ ſint æquales, & tertia ſigni contrarii, figura erit *Cruciformis* atque *ſpeciei* 18 vel 19.

V. Quod ſi $4c^2 - \frac{2c \times \overline{a-c} \times \overline{a-b}}{b} = \frac{\overline{a-c}^2 \times \overline{a-b}^2}{b^2}$ Curva habebit tres diametros atque erit *ſpeciei* 30.

VI. Hujuſque ſuppoſuimus BQ poni inter C & S; at ſi ad alias partes vel puncti C vel S transferatur, Curva unicam habebit Aſſymptoton perpendicularem ipſi CS ad punctum D, ita ut CD : CB : : SA : SB; æquatio reducta ad formam D. *Neutoni* in hoc caſu eſt hujuſmodi

$$xy^2 = -\frac{\overline{b \mp a}}{b}x^3 - \frac{3cb \pm \overline{a+c} \times \overline{b \mp a}}{b}x^2$$

$$-\frac{\overline{3bc^2}}{b \mp a} + \overline{2c \pm a} \times x - \frac{b^2 c^3}{\overline{b \mp a}^3} + bc^2 \frac{\overline{c \pm a}}{b \mp a}.$$

Cujus Radices ſunt $\frac{cb}{b \mp a}$, $\frac{cb}{b \mp a}$, $\frac{cb}{b \mp a} \mp a - c$.

Signa

Signa ſuperiora obtinent cum AN ponitur ad alias partes puncti S quam eſt C; Terminus x^3 eſt negativus & deeſt terminus y, adeoque curvæ pertinent ad Hyperbolas defectivas diametrum habentes §. 20. in Enumeratione *Neutoniana*; quare ſi A ſit inter C & S (quoniam in eo caſu a eſt major quam c, adeoque $\frac{cb}{b+a}+a-c$ major quam $\frac{cb}{b+a}$) patet duas radices minores fore æquales & tertiam ejuſdem ſigni, adeoque curvam eſſe *ſpeciei Quadrageſimæ tertiæ*, quæque *Nodata* eſt in C & Ovalem habet inter C & A jacentem.

VII. Sit AN ad alias partes puncti S quam eſt C; eruntque radices omnes ejuſdem ſigni ſi SA ſit major SB; ſed cum $\frac{cb}{b-a}$ major ſit quam $\frac{cb}{b-a}-c-a$, patet in hoc caſu duas majores radices æquari, & proinde curvam eſſe *ſpeciei Quadrageſimæ tertiæ*, & Hyperbolam eſſe Conchoidalem habentem *Punctum* Conjugatum in C ad Convexitatem ſuam.

VIII. Quod ſi SB major ſit SA, adeoque $\frac{ac}{b-a}$ minor quam a, patet radicem tertiam $\frac{cb}{b-a}-a-c=\frac{ac}{b-a}-a$ reddi negativam; proinde in hoc caſu radices duæ ſunt æquales, & tertia ſigni contrarii, adeoque curva erit *ſpeciei Quadrageſimæ quartæ*, & habebit *Punctum* Conjugatum ad Concavitatem ſuam.

IX. Quod ſi denique AN tranſeat per C, radices omnes $\frac{ab}{b+a}$, $\frac{ab}{b+a}$, $\frac{ab}{b+a}$ erunt æquales, & curva ſit *Cuſpidata* in C, atque evadit *Ciſſois* veterum. Quæ ſpecies eſt *Quadrageſima ſecunda*.

X. Hucuſque eas ſolas curvas deſcripſimus quæ diametrum habent; aliæ vero quæ diametris carent facillime deſcribuntur, ſi recta BQ ut prius maneat perpendicularis in CS, recta vero (*Fig.* 24.) AN ei ſit parallela. Sit CB $=b$, BA $=c$, CS $=a$, PM $=y$, CM $=x$, & æquatio curvæ prodibit facili calculo hujus formæ $\mp b^2y^3+b^2cy^2=\overline{a\pm b}\times bx^2y+ab\times\overline{a\pm b}\times xy-c\times\overline{a+b}^2\times x^2$ quæ ad formam *Neutonianam* ſic reduci poteſt. Ad F punctum medium inter C & S erigatur FK normalis in CS; atque in ea ſumatur FD quæ ſit ad BA ut SB ad CB; ducatur per D recta DH parallela rectæ CS, & producatur PM ad R; dicatur nunc DR $=y$, PR $=x$, & æquatio prodibit formæ *Neutonianæ*

$$xy^2 + \frac{\overline{a-b}\times ca}{b}y = \frac{bx^3}{a-b} - \frac{\overline{3c-bc}}{a-b}\times x^2 + \overline{\frac{3\times\overline{a-b}\times c^2}{b}+2c^2+\frac{a^2}{4}}\times x$$

$$-\frac{\overline{a-b}^2\times c^3}{b^2} - \frac{\overline{a-b}\,c^3}{b} - \frac{a^2c\times\overline{a-b}}{4b}.$$

Inveſtigentur radices æquationis $\frac{bx^4}{a-b} - \frac{\overline{3c-bc}}{a-b}\times x^3 + \overline{\frac{3\times\overline{a-b}\times c^2}{b}+2c^2+\frac{a^2}{4}}\times x^2 - \overline{\frac{\overline{a-b}^2\times c^3}{b^2}-\frac{\overline{a-b}\times c^3}{b}-\frac{a^2c\times\overline{a-b}}{4b}}\times x + \frac{\overline{a-b}^2\times c^2a^2}{4b^2} = 0$ ſecundum præcepta D. *Neutoni*; & prodibunt $\frac{c\times\overline{a-b}}{b}$, $\frac{c\times\overline{a-b}}{b}$, $\frac{a}{2b}\times c+\sqrt{c^2-ab+b^2}$, $\frac{a}{2b}\times c-\sqrt{c^2-ab+b^2}$.

Si BA minor ſit quam $\frac{CS}{2}$, ſeu c minor quam $\frac{a}{2}$, & ſimul ſi (ducto ſemicirculo ſuper CS rectam AN ſecante in E, & perpendiculo EG in CS) ſit CG major CB, erunt omnes radices reales; atque duæ æquales erunt vel majores reliquis duabus vel iis utriſque minores, & curva conſtabit ex tribus Hyperbolis cum Ovali; eritque *Nodata* in C & *ſpeciei ſecundæ*.

XI. Quod ſi CG = CB, & Circulus ſuper diametrum CS deſcriptus tranſeat per A, curva non erit Linea Ordinis tertii ſed Conica ſectio; nam tunc ſit CB : BA :: BA : SB & $c^2+b^2=ab$ adeoque eſt $\sqrt{c^2-ab+b^2}=0$ & radices duæ ultimæ evadunt quoque æquales.

XII. Quod ſi ſit CB major CG, neque tamen æqualis rectæ SG, erit c^2+b^2 minor ab adeoque c^2-ab+b^2 evadet Quantitas negativa cujus radix eſt impoſſibilis; proinde duæ ultimæ radices fiunt impoſſibiles, aliæ vero duæ ſunt æquales, adeoque figura erit *Cruciformis ſpeciei ſeptimæ* vel *octavæ*.

XIII. Quod ſi BA $=\frac{CS}{2}$, ſeu $c=\frac{a}{2}$, erit $c^2-ab+b^2=\frac{a^2}{4}-ab+b^2$, & $\sqrt{c^2-ab+b^2}=\frac{a}{2}-b$; adeoque radices erunt $\frac{a\times\overline{a-b}}{2b}$, $\frac{a\times\overline{a-b}}{2b}$, $\frac{a\times\overline{a-b}}{2b}$, $\frac{a}{2}$. Et proinde tres radices æquantur, adeoque curva evadit *Cuſpidata* in C *ſpeciei tertiæ*.

XIV. Si

XIV. Si vero BA ſit major quam $\frac{CS}{2}$, radices mediæ erunt æquales, & figura conſtabit ex tribus Hyperbolis ut prius cum *Puncto* conjugato in C; eritque curva *ſpeciei quartæ*.

XV. Hucuſque ſuppoſuimus rectam BQ poni inter C & S: Quod ſi transferatur ad alias partes puncti C quam eſt S, vel ad alias partes puncti S quam eſt C; curva evadet Hyperbola defectiva unicam habens Aſſymptoton, & duo tantum crura Hyperbolica juxta Aſſymptoton illam in plagas oppoſitas progredientia. Quod ſi BA ſit minor quam $\frac{CS}{2}$, curva habebit *Nodum* in C cum Ovali, eritque *ſpeciei* 34.

XVI. Quod ſi BA $= \frac{CS}{2}$ curva evadet *Cuſpidata* in C atque *ſpeciei* 35.

XVII. Denique ſi in hoc caſu BA ſit major quam $\frac{CS}{2}$; curva habebit *Punctum* conjugatum in C evaneſcente nodo & ovali; eritque curva *ſpeciei* 36. Hæc omnia vel ex Cl. *Neutoni* Tractatu, vel ex *Corol.* 4. *Prop.* 7. facile deducuntur.

Curvas omnes pure Hyperbolicas tertii Ordinis quæ punctum duplex habent ad diſtantiam finitam deſcripſimus. Reſtant curvæ Hyperbolo-Parabolicæ & pure Parabolicæ, quarum Deſcriptiones facillimæ ex methodo ipſius *Prop.* 5. deduci poſſunt.

XVIII. Sint in *Prop.* 5. (*Fig.* 25.) Anguli SNP, FCO recti; ſitque BQ perpendicularis in CS, & AN eidem parallela; ſit CB $= b$, BA $= c$, CG $= a$, PM $= y$, CM $= x$, & æquatio curvæ erit

$$bx^2y + \overline{a-b} \times bxy - \frac{b^2c}{a}x^2 = \frac{\overline{a-b}^2 \times c}{a}y^2.$$ Sumatur igitur CD $= \frac{SB}{2}$, atque DF ad BA, ut eſt BC ad CS, & æquatio ad axem FB (ſi F ſumatur principium abſciſſæ) erit formæ *Neutonianæ*

$$y^2x + \frac{\overline{a-b} \times bc}{a}y = \frac{\overline{a-b}^2 \times c}{ab}x^2 + \overline{\frac{2 \times \overline{a-b}^2 \times c^2}{a^2} + \frac{\overline{a-b}^2}{4}} \times x$$
$$+ \frac{\overline{a-b}^2 \times c^3 b}{a^3} + \frac{\overline{a-b}^2 \times cb}{2a}.$$ Radices vero Æquationis cubicæ

$$\frac{cx^3}{ab} + \frac{\overline{8c^2 + a^2}}{4a^2} \times x^2 + \overline{\frac{c^3b}{a^3} + \frac{bc}{2a}} \times x + \frac{b^2c^2}{4a^2}$$ ſecundum præcepta D. *Neutoni* ſunt inveſtiganda, & prodeunt quidem $\frac{bc}{a}$, $\frac{bc}{a}$, $\frac{ab}{4c}$. Unde ſi BA

ſit minor quam $\frac{CS}{2}$, curva habebit *Nodum* in C eritque *ſpeciei Quadrageſimæ ſeptimæ.*

XIX. Quod ſi $BA = \frac{CS}{2}$, erunt radices tres æquales ipſi $\frac{b}{2}$; & proinde curva evadit *Cuſpidata* in C *ſpeciei Quadrageſimæ octavæ.*

XX. Si vero BA major ſit quam $\frac{CS}{2}$, radices æquales $\frac{bc}{a}$, $\frac{bc}{a}$ fiunt majores tertia, & curva fit *Punctata ſpeciei Quadrageſimæ nonæ.*

XXI. Sint nunc AN & BQ parallelæ ſibi mutuo, atque rectæ CS perpendiculares; ſitq; $CB = b$, $SA = c$ & $CS = a$ eritq; ſi $CM = x$, $PM = y$ $y^2x + \frac{\overline{a-c}\times\overline{a-b}}{a}xy^2 = \frac{cb^2}{a\times\overline{a-b}}x^2$. Proinde ſi ſumatur CD ad CB, ut SA ad CS, & D ſit principium abſciſſæ, ſeu $DM = x$. Æquatio prodibit formæ D. *Neutoni* $xy^2 = \frac{cb^2}{\overline{a-b}\times a}x^2 - \frac{2cb^2}{a^2}\overline{a-c}\times x$ $+ \frac{cb^2\,\overline{a-b}\times\overline{a-c}^2}{a^3}$, cumque æquationis $x^2 - \frac{2\times\overline{a-c}\times\overline{a-b}}{a}x$ $+ \frac{\overline{a-b}^2\times\overline{a-c}^2}{a^2}$ radices duæ $\frac{\overline{a-b}\times\overline{a-c}}{a}$, $\frac{\overline{a-b}\times\overline{a-c}}{a}$ ſint æquales, patet curvam eſſe *ſpeciei Quadrageſimæ tertiæ* & Hyperbolo-Parabolicas lineas ſe mutuo decuſſare ad morem *Crucis* in C.

XXII. Sint porro rectæ AN & BQ (*Fig.* 26.) parallelæ ipſi CS; ſitque $CB = 2CA$, $CB = b$, $CS = a$, atque æquatio curvæ ſi $CM = x$, $PM = y$ erit $y^2 = \frac{2ax^3}{b^2} + \frac{a^2 - b^2}{b^2}\times x^2$. Unde ſi CB ſit minor quam CS curva habebit *Nodum* in C cum Ovali eritque Parabola *ſpeciei* 68.

XXIII. Quod ſi $CB = CS$, æquatio curvæ erit $y^2 = \frac{2x^3}{b}$ unde curva evadit *Cuſpidata* in C atque ipſa eſt Parabola Neiliana ſeu ſemicubica *ſpeciei* 70.

XXIV. Si vero CB major ſit quam CS, curva erit *Punctata* in C *ſpeciei Sexageſimæ nonæ.*

Hæ ſimpliciſſimæ ſunt Conſtructiones omnium fere Linearum tertii Ordinis, quæ aliquod punctum duplex habent; atque ſingulæ duplici ratione demonſtrantur; vel ex *Corol.* 4. *Prop.* 8. vel ex ipſis D. *Neutoni* Characteribus quibus curvarum ſpecies diſcriminavit.

Schol. II. Simili ratione qua in hac *Prop.* Curvarum *Prop.* 5. & 8. deſcriptarum Reductionem ad *Neutonianas* demonſtravimus, oſtendi poteſt æquationes omnes trium dimenſionum, ad quatuor *Neutonianas* reduci

duci poſſe. Quippe omnis Linea tertii Ordinis aliquam habet Aſſymptoton, & in ejus æquatione ordinatæ illi Aſſymptoto parallelæ ad duas tantum dimenſiones aſcendere poſſunt; omnes proinde æquationes Linearum tertii Ordinis ad hanc poſſunt reduci $ax^3 + bx^2y + cxy^2 + dx^2 + exy + fy^2 + gx + hy + k = o$, terminus vero cx^2y exterminari poteſt ut in Propoſitione docuimus; termini quoque exy, fy^2, facile exterminantur ut in pluribus exemplis oſtendimus, unde manebunt tantum termini xy^2, y, x^3, x^2, x, & k; atque nonnunquam loco xy^2, y habebimus tantum y^2, xy vel ipſum y; ut ſatis eſt manifeſtum.

Prop. IX.

Si in Prop. 5. *quando crus* NL *anguli* SNP *tranſit per punctum duplex* C, *crus* CO *anguli* FCO *ſimul coincidat cum* NP; *curva deſcripta non erit Linea tertii Ordinis ſed Sectio Conica.*

Ex *Prop.* 7. manifeſtum eſt omnes curvas quas hac *Prop.* aſſerimus eſſe Conicas ſectiones, eas eſſe quæ *(Fig. 27.)* ſecundum *Prop.* 7. facilius deſcribuntur, ſi circulus SQC (habens angulum SNL inſcriptum) & rectæ BQ, AN in eodem puncto D concurrant. Eas vero ſectiones eſſe Conicas ſic demonſtratur.

Jungatur CD, ſint PM, QT, CB parallelæ & DCB = SNL, atque ad rectam CD inveſtigetur æquatio curvæ, dicatur CB $= b$, CA $= c$, CD $= d$, DS $= a$, PM $= y$, CM $= x$, eritque TQ $= \frac{dby}{dy + bx}$, CT $= \frac{dbx}{dy + bx}$. Sed AU : UN : : $c : d$ & DT : SD + QT (= QK) : : SH (= $a + c$ — AU) : NH = CD — NU, unde prodit NH $= \frac{ad^2y}{d \times \overline{a + b - c} \times y + abx}$. Sit NHR = NRH ſitque NH : HR : : $a : f$, atque erit SR $= \frac{ad \times \overline{a + b} \times y + a^2bx + fd^2y}{\overline{a + b - c} \times dy + abx}$. Similia vero ſunt triangula SRN & NZP, adeoque SR : NR : : NZ : PZ. Unde $ad \times \overline{a + b} \times y + a^2bx - fd^2y : \overline{a + b - c} \times dyx + abx^2 + \overline{c - b} \times d^2y + adbx :: d^2ay : \overline{a + b - c} \times dy^2 + abyx - dcay$. Proinde erit $\overline{a + b} \times ady$ $+ a^2bx$

$+a^2bx - fd^2y : \overline{a+b-c} \times dyx + abx^2 + \overline{c-b} \times d^2y + adbx :: d^2a : \overline{a+b-c} \times dy + abx - dca$. Unde ducendo media & extrema in ſe mutuo prodibit æquatio duarum tantum dimenſionum; quæ proinde erit ad ſectionem Conicam.

Corol. I. Punctum C nunc evadit ſimplex, & cum curva non tranſeat per S, hinc poſtea hæc Deſcriptio erit nobis uſui, cum ſectio Conica eſt deſcribenda per punctum C, quæ non tranſit per S; nam in deſcriptione *Prop.* 1. curva tranſit per utrumque punctum.

Corol. II. Quoniam in Conſtructione hujus *Prop.* BQ neceſſario ſemper circulo occurrat bis vel eum contingat, patet hac ratione eas ſolas ſectiones Conicas deſcribi quæ crura habent infinita; Ovales enim hac ratione deſcribi nequeunt.

Satis explicuimus methodum *Prop.* 5. qua Lineæ tertii Ordinis punctum duplex habentes deſcribi poſſunt; reſtant duæ aliæ methodi eaſdem delineandi quas pauciſſimis explicabimus.

Prop. X.

Cæteris manentibus ut in Prop. 5. *ducatur Concurſus crurum* NL *&* CF (Fig. 28.) *per rectam infinitam* BQ *dico Concurſum crurum* CO *&* SN *deſcripturum Lineam tertii Ordinis.*

Sint PM, NU, QK Normales in CS, atque anguli FHC, PRC æquales ſibi mutuo atque angulo dato FCO, angulus vero NTS=SNL, & ſit Sg perpendicularis in NT ac Qe parallela CS, dicatur recta $CS = a$, $CB = b$, $SA = g$, $PM = y$, $SM = x$, $CM = a - x$; ſit ſinus anguli FCO ad coſinum ut d ad a; ſit BK ad KQ, ut a ad c, & Au ad Nu ut a ad e. His poſitis eadem plane ratione atque in *Prop.* 5. deprehendes $QK = \frac{bc \times \overline{ay + d \times \overline{a-x}}}{\overline{dc + a^2} \times y - \overline{a \times c - d} \times \overline{a - x}}$ & $CK = \frac{bdcy - bac \times \overline{a-x}}{\overline{dc + a^2} \times y - \overline{a \times c - d} \times \overline{a - x}}$; cumque SM : PM :: SU : NU; & ſit $SU = g - \frac{a\text{NU}}{e}$, erit $NU = \frac{gy}{x + \frac{a}{e}y}$ & $SU = \frac{gx}{x + \frac{a}{e}y}$. Sed ſimilia ſunt triangula SgN & NeQ,

adeoque

adeoque $Sg : gN :: Ne : Qe$, & ſimilia quoque ſunt triangula SgT & Nyg, adeoque $Sg : gN :: ST : Ny$; unde $ST : Ny :: Ne : Qe$. Sed $ST = SU + UT =$ (ſi ſit ſinus anguli SNP ad coſinum ut a ad f) $SU + \frac{fNU}{a}$,

$$= \frac{gx + \frac{fgy}{a}}{x + \frac{ay}{e}} \;\&\; Ny = NU - Uy = \frac{gy - \frac{fgx}{a}}{x + \frac{ay}{e}} \;\&\; Ne = NU - QK$$

$$= \frac{\overline{dc + a^2} \times gy^2 - ga \times \overline{c - d} \times \overline{a - x} \times y - bc \times \overline{ay + da - dx} \times \overline{x + \frac{ay}{e}}}{\overline{dc + a^2} \times y - a \times \overline{c - d} \times \overline{a - x} \times x + \frac{ay}{e}},$$

& $Qe = a + CR - SU$. Unde deducitur hæc Analogia

$$x + \frac{fy}{a} : y - \frac{fx}{a} :: g \times \overline{cd + a^2} \times y^2 - ga \times \overline{c - d} \times \overline{a - x} \times y - bcayx - \frac{bca^2y^2}{e}$$

$$- bcd \times \overline{a - x} \times x - bcd \times \overline{a - x} \times \frac{ay}{e} : \overline{dc + a^2} \times ayx + \frac{a^2}{e} \times \overline{dc + a^2} \times y^2$$

$$- a^2 \times \overline{c - d} \times \overline{a - x} \times x - \overline{c - d} \times \frac{a^3}{e} \times \overline{a - x} \times y + bdcyx + \frac{bcda}{e} \times y^2$$

$$- bac \times \overline{a - x} \times x - \frac{ba^2c}{e} \times \overline{a - x} \times y - g \times \overline{dc + a^2} \times yx + ga \times \overline{c - d} \times \overline{a - x} \times x.$$

Unde ducendo media & extrema in ſe mutuo prodibit æquatio hujus formæ ad Axem CS. $Ay^3 + By^2x + Cyx^2 + Dx^3 + Ey^2 + Fxy + Gx^2 = o$, Coefficientes vero A, B, C, D, E, F, G ex hac Analogia obvio calculo inveſtigari poſſunt, & ſpeciatim D (coefficiens Termini x^3) $= -\frac{fcdb}{a} - bac + \overline{ga - a^2} \times \overline{c - d}$ & G (coefficiens Termini x^2) $= aD$.

Cum vero in hac æquatione x & y ad tres tantum aſcendant dimenſiones, manifeſtum eſt curvam eſſe Lineam Ordinis tertii, & in tribus punctis rectâ poſſe ſecari.

Corol. I. Curva habet punctum duplex in S; quippe evaneſcente y, æquatio evadit hujus formæ $Dx^3 + Gx^2 = o$; ſed $G = aD$, adeoque æquationis tres radices ſunt o, o, a. Unde curva tranſit per S bis & ſemel per C; aſt in nullo alio puncto præter hæc duo curva ſecat rectam CS.

Corol. II. Sit angulus DCS = FCO, & ſuper DS deſcribatur Arcus Circuli cui inſcribi poſſit datus angulus SNP; ſi recta AN huic circulo bis occurrat, Curva habebit punctum duplex & *Nodum* in S; quod ſi recta AN circulum contingat Ovalis evaneſcet & curva evadet *Cuſpidata* in

in S; quod ſi Recta AN tota ponatur extra circulum curva habebit *Punctum* conjugatum in S abſque nodo, Cuſpide vel Ovali.

Corol. III. Quod ſi curva deſcribi ſupponatur concurſu cruris CO cum SR angulum datum RSN conſtituente cum SN, curva ejuſdem Ordinis deſcribetur ac in *Prop.* quippe hæc conſtructio ad eam Propoſitionis reduci poteſt, cum SR perpetuo occurrat rectæ NP in puncto rectæ RX, ex *Lemmate* 1. Poſitione datæ, & per idem *Lemma* angulus SRP detur.

Corol. IV. Concurrant rectæ BQ & AN (*Fig.* 29.) in E, & ducatur EG ita ut GEB = SNL, & ſumatur SGH æqualis ipſi BEN, atque cæteris manentibus ut in *Prop.* ſubſtituatur EG loco rectæ AN, & angulus SGH loco anguli SNL, eademque plane curva deſcribetur quæ in Propoſitione. Concurrant enim rectæ GH & NL in T, atque angulus GTN = SGH — SNT = BEN — BEG = GEN, adeoque circulus per G, N, E ductus tranſibit per T; ſed ex hypotheſi QEG = QNG, adeoque circulus per G, N, E ductus tranſibit per Q: ſed Q, T, N ſunt puncta in eadem recta Linea, adeoque Q & T coincidunt, proinde BT, BQ atque CT ac CQ etiam coincidunt, unde eadem curva deſcribetur ſi angulus SGT percurrat rectam EG, vel ſi SNL percurrat rectam AN.

Corol. V. Jungatur SE (*Fig.* 30.) & ſi angulus SNL æqualis ſit angulo SEB, curva non erit Linea tertii Ordinis ſed ſectio Conica. Quippe cum SEQ = SNQ, circulus per S, N, E deſcriptus tranſibit per Q, & proinde angulus QSN dabitur; nam æqualis erit ſupplemento anguli BEN ad duos rectos. Duo igitur anguli dati QSN, QCO revolvuntur circa data puncta C & S, & crurum CQ, SQ concurſus ducitur per rectam BE; proinde per *Prop.* 1. crurum CO & SN concurſus deſcribet ſectionem Conicam.

Prop. XI.

Cæteris manentibus ut in Prop. X. *ſi* CP *&* SP *ſimul coincidant cum* CS *curva evadet ſectio Conica.*

COincidat SP cum SC, (*Fig.* 31.) & perveniat N ad A, dico ſi eodem tempore coincidat CO cum CS Lineam deſcribi ſecundi Ordinis. Ducatur CK normalis ad CS, & producatur NQ donec ei occurrat in K; ſit H punctum ubi BQ eidem normali occurrit, atque angulus SNL = GQK, FCO = CQG atque CBQ = GQH. Proinde iiſdem

iiſdem retentis ſymbolis ac in *Prop*. præcedente, erit QG : GH :: $a : c$, & QG : GC :: $a : d$ & QG : GK :: $f : a$; adeoque GH : GC :: $c : d$; & CH : GC :: $d - c : d$; ſimiliter GK : GC :: $a^2 : df$, & CK : GC :: $a^2 + df : df$, & proinde CH : CK :: $\overline{d-c} \times f : a^2 + df$. Porro CH : CB :: $c : a$, & CH $= \frac{bc}{a}$; ſimiliter CK : CA :: $a : f$, & CK $= \overline{g-a} \times \frac{a}{f}$; Unde $\overline{g-a} \times \frac{a}{f} : \frac{bc}{a} : a^2 + df : \overline{d-c} \times f$ & $\overline{g-a} \times a^2 \times \overline{d-c} = \overline{a^2 + df} \times bc$, adeoque $fcdb + ba^2c + a^3c - a^3d - ga^2 \times \overline{c-d} = 0$. Cum vero hæc ipſa Quantitas ſit coefficiens termini x^2, & ea diviſa per a ſit coefficiens termini x^3, manifeſtum eſt duos hos Terminos evaneſcere, & proinde nullum in æquatione Terminum manere in quo non reperiatur y; proinde omnes æquationis Termini per y dividantur, & æquatio manebit duarum tantum dimenſionum, adeoque erit ad Conicam ſectionem.

Corol. I. Proinde ſi ad punctum A ſuper recta SA, conſtituatur angulus SNK = SNL, & ad punctum C ſuper eadem recta angulus ACE = FCO, ſi NK & EC in eodem puncto Q occurrant rectæ BQ, Linea deſcripta erit ſectio Conica.

Corol. II. Hinc ſi anguli SNL, FCO (*Fig.* 32.) conficiant duos rectos, & ſi angulus ABQ = FCO, deſcribetur ſectio Conica; atque ſi anguli SNL, FCO ſint recti, & BQ ſit perpendicularis in CS, Linea deſcripta erit ſectio Conica non tertii Ordinis.

Corol. III. Curva deſcripta tranſit ſemel per S, ſed punctum C non attingit.

Ad Aſſymptotos curvarum *Prop*. 10. deſcriptarum inveſtigandas (quod Problema altioris eſt generis quam *Prop*. 6.) *Lemma* 2. eſt præmittendum.

Lemma II.

Dentur puncta C *&* S (Fig. 33.) *recta* AN, *& anguli* SNL, CPN; *percurrat punctum angulare* N *rectam* AN, *& punctum* P *interea deſcribet Hyperbolam defectivam tertii Ordinis, quæ unicam habet Aſſymptoton, eſtque ſpeciei* 34, 35, 36, 41, 42, 43 *vel* 44.

Ad *Lemma* hoc facilius domonſtrandum imprimis oſtendemus eandem curvam angulis obliquis quibuſcumque SNL & CPN atque rectis deſcribi poſſe; cum facilius in caſu quo anguli ſunt recti æquatio inveſtigari poſſit & prodeat ſimplicior.

Ex dato puncto S ducatur SR parallela ipſi CP, & quoniam dantur anguli CPL, SNL, dabitur eorum differentia NSR; proinde per *Lemma* 1. recta SR occurret NL productæ in puncto R, quod tangit rectam

F AR

AR conſtituentem angulum RAN = RSN ad punctum A ubi angulus SAN = CPL.

Sit SB perpendicularis in CS, & ſumatur angulus BCS æqualis ſupplemento dati CPL ad rectum; ducatur RQ parallela rectæ SB, & huic in Q occurrat PQ normalis in CP ad punctum P; ducatur RE parallela ipſi CS & anguli QRE, QPE erunt recti, adeoque circulus per puncta R, Q, E ductus tranſibit per P, & angulus REQ = RPQ = BCS; unde cum RE = CS, erit RQ = SB; adeoque cum RQ & SB ſint æquales & parallelæ, erunt SR & BQ parallelæ, adeoque BQ & CP parallelæ erunt, & proinde angulus BQP erit rectus. Sed cum RQ ſit invariatæ magnitudinis & ſemper parallela ipſi SB, patet Q ſemper reperiri in recta parallela ipſi AR; angulus igitur BQP angulari puncto Q rectam percurrit, & CP in PQ eſt Normalis, unde patet eandem curvam deſcribi his angulis rectis, quæ prius angulis obliquis SNL & CPL deſcribebatur.

Rectam igitur infinitam AE *(Fig.* 34.*)* percurrat angulari ſuo puncto N angulus rectus SNL, & in crus NL demittatur ex dato puncto C Normalis CP, atque P erit in curva quam Hyperbolam eſſe defectivam in *Lemmate* aſſerimus. Dicatur CM $= x$, PM $= y$, AB $= b$, CB $= a$, SA $= c$; cumque ſimilia ſint triangula CMP, RPN, SAN, erit CM (x) : PM (y) :: NA : SA (c) unde NA $= \frac{cx}{y}$, atque CM (x) : PM (y) :: RP $(b-y)$: RN $= \frac{by-y^2}{x}$. Unde cum RN + NA $= a + x$ erit $\frac{cx}{y} + \frac{by-y^2}{x} = a + x$, & $by^2 - y^3 = x^2y + axy - cx^2$. Sit F punctum medium inter C & B & ad F erigatur Linea perpendicularis in CS, in qua ſumatur DF = SA, & ducatur DH parallela rectæ AE, eritque DH curvæ Aſſymptotos, & ſi DT (= HP) ſit nunc x atque PT (= DH) ſit y Æquatio erit formæ *Neutonianæ*

$$y^2x - cay = -x^3 + \overline{3c-b}\times x^2 - \overline{3c^2 + 2cb + \tfrac{a^2}{4}}\times x + c^3 - bc^2 - \frac{a^2c}{4}.$$

Ubi quoniam x^3 eſt negativus manifeſtum eſt curvam eſſe Hyperbolam defectivam unicam habentem Aſſymptoton & duo crura Hyperbolica in contrarias plagas progredientia. Si ſemicirculus ſuper CS deſcriptus bis occurrat rectæ AE curva erit *ſpeciei* 34; ſi recta Circulum contingat curva erit *ſpeciei* 35; quod ſi extra Circulum prorſus cadat curva erit *ſpeciei* 36; in primo caſu curva eſt *Nodata*, in 2° *Cuſpidata*, in 3° *Punctata*.

Si C & S ſint in eadem recta Normali ad AE evadet CB $= a = 0$, adeoque evaneſcet terminus cay, & curva diametrum habebit. Si A ſit inter C & S curva habebit *Nodum* in C, & Ovalem inter C & A, eritque

que *ſpeciei* 41; Quod ſi ſit C in A, evaneſcit Ovalis & curva fit *Cuſpidata* in C *ſpeciei* 42, eſtque *Ciſſois* veterum. Quod ſi ſit C inter S & A curva evadet Conchoidalis cum *Puncto* conjugato ad Convexitatem eritque *ſpeciei* 43. Denique ſi ſit S inter C & A *Punctum* erit ad Concavitatem Conchoidalis & curva erit *ſpeciei* 44.

Hæ ſimpliciſſimæ ſunt Lineæ tertii Ordinis & commodiſſime deſcribuntur tum methodo D. *Neutoni* poſtea demonſtranda tum noſtra. Quapropter nonnullas earum ſimpliciores proprietates non pigebit enumerare ſequentibus *Corolariis*.

Corol. I. Ducatur SK parallela datæ AN & CK ei perpendicularis in K. Sumatur AB = CK (*Fig.* 35.) & ducatur BD parallela ipſi AN; ſuper CS deſcribatur ſemicirculus CRS, & ex C educatur recta quævis CQ occurrens datæ rectæ BD in Q & circulo in R, ſume QP = CR, & Q*p* = CR, eruntque P & *p* puncta in curva hujus *Lemmatis*; quippe QN = CS, quoniam CB = KA, & CQ = SN ex hypotheſi, & SR eſt æqualis ac parallela ipſi PN, atque angulus SRC = QPN, cum ſint recti & proinde erit QP = CR; hinc vero infinita curvæ puncta facillime reperiuntur.

Corol. II. Quod ſi loco ſemicirculi deſcribatur Arcus quicunque circularis (*Fig.* 35. *n.* 2.) tranſiens per C & S: & ad circulum illum educatur CR rectæ datæ BD occurrens in Q, & ſumatur ut prius QP = CR; curva deſcripta ea erit quæ deſcribitur, ſecundum *Lemma* hoc, motu anguli SNL æqualis ei qui in circulo dicto inſcribi poteſt (angulo nimirum SRC) adeoque coincidet cum iis *Corolario* præcedente deſcriptis ipſius ſemicirculi ope.

Corol. III. Super recta CS (*Fig.* 36.) deſcribatur ſemicirculus CQS, & ex C in AN ſit CD normalis; dico ſi duo anguli recti ECH, ODK moveantur circa C & D ut Polos & Concurſus crurum duorum CE, DO ducatur per dictum ſemicirculum concurſum reliquorum crurum CH & DK, deſcripturum eandem curvam *Lemmate* & *Corol.* 1. alia ratione delineatam. Quippe cum angulus in ſemicirculo CQS rectus ſit, atque anguli (in Deſcriptione *Lemmatis*) CPN, PNQ & CDN ſint quoque recti, patet circulum eundem per quinque puncta C,P,Q, N & D poſſe tranſire; aſt angulus PNQ eſt rectus adeoque rectus erit PDQ; proinde rectorum angulorum QDP, QCP crura CQ & DQ concurrunt in ſemicirculo, & crurum CP ac DP concurſus curvæ *Lemmate* deſcriptæ punctum P ſemper denotat. Si circulus ſecet rectam AN curva erit *ſpeciei* 34, ſi eam contingat 35, ſi vero nec ſecet nec circulum contingat curva erit *ſpeciei* 36. Hæc ita ſe habent quoties CS eſt obliqua ad AN, ſed ſi A & D coincidant, curva habebit diametrum eritque *ſpeciei* 41, 43 aut 44, ſecundum varias poſitiones rectæ AN.

Corol. IV. Quod ſi angulus SNP ſit obliquus quicunque datus & (*Fig.* 36. *n.*2.) CPN quoque obliquus; duc CD angulum CDE conſtituens æqualem dato CPN. & circa datum punctum D moveatur angulus QDP = QNP, & circa C moveatur PCQ æqualis ſupplemento anguli QDP ad duos rectos, ducatur CQ & DQ per circulum cui inſcribi poteſt angulus SQC æqualis ſupplemento dati CPN ad duos rectos, & Concurſus crurum CP, DP deſcribet Lineam terti Ordinis quam præcedente *Corolario* Angulorum rectorum ope deſcripſimus. Hujus demonſtratio ſimilis eſt præcedenti.

Corol. I. Coincidat D cum centro (*Fig.* 37.) Circuli & ſit angulus CDN = QCP = SNP. Ducatur per C recta CR parallela datæ DN; cumque PCQ ſupplementum ſit anguli QDP ad duos rectos erit CQD, ſupplementum anguli CPD ad totidem rectos, & proinde erit CQD = CPR; ſed DQC = DCQ ex natura circuli, & DCQ = PCR (cum DCR = QCP) adeoque CPR = PCR, & triangulum PCR erit Iſoſceles; unde PR = RC. Proinde ſi (*Fig.* 38.) ex dato puncto D, ad datam rectam quamvis CR, educatur DC conſtituens angulum DCR æqualem dato SNP, & ex eodem puncto D aliæ rectæ infinitæ ad CR educantur ut DR, & ſemper ſumatur RP = RC, & RC = R*p* curvarum aliquæ de quibus agimus, quas præcedentibus *Corolariis* diverſimode deſcripſimus, hac facili ratione conſtrui poterint.

Corol. VI. Angulus (*Fig.* 37.) SQC=PNQ=CDA; ſed $SQC = \frac{SDC}{2}$, adeoque CDA erit dimidium anguli CDS, & proinde erit AN perpendicularis in CS, & A erit punctum medium inter C & S; unde quoties AN perpendiculariter inſiſtit medio rectæ CS, & anguli SNP & CPN ſunt æquales, facilior hæc conſtructio obtinebit. Ex *Lemmate* manifeſtum eſt unicam hujus curvæ Aſſymptoton eſſe parallelam rectæ CR; (*Fig.* 38.) ſed ſi ſint PM, *pm* parallelæ ipſi CR, erit CM = C*m*, quoniam RP = R*p*; adeoque quando R abit in infinitum, & coincidit P cum D, & *p* abit infinitum, erit CM = CD, & C*m* = CF = CD: Proinde ſi per F ducatur FG parallela ipſi CR ea erit Aſſymptotos curvæ. Quoties recta CD (*Fig.*37.) ſit perpendicularis in AN & A ac D coincidunt curva habet CD diametrum; ſed ſi CD ſit Obliqua in CR curva non habebit diametrum; in primo caſu curva deſcribitur *ſpeciei* 41, in ſecundo *ſpeciei* 34.

Corol. VII. Ultimæ huic ſimplici curvarum conſtructioni Analoga eſt quam inveni curvarum hujus claſſis etiam puncto duplice carentium deſcriptio. Dentur in (*Fig.* 39.) eodem plano puncta duo C & D atque recta AR; ad ſingula rectæ AR puncta ducantur ex C & D rectæ CR & DR, & in recta DR ſemper capiatur RP = RC, & R*p* = RC, ac deſcribetur curva tertii Ordinis puncto duplice deſtituta, ſi neutrum punctorum C & D exiſtat in recta AR. Quippe dicatur DB = *a*, CA = *b*, BA = *c*,

$BA = c$, $PM = y$, $DM = x$, & cum $DM : PM :: DB : BR$, erit $BR = \frac{ay}{x}$, $AR = c + \frac{ay}{x}$, unde $CR^2 = \overline{c + \frac{ay}{x}}^2 + b^2$ & $PR^2 = \overline{\frac{ay}{x} - y}^2 + \overline{a - x}^2$ ſed $CR = PR$, unde $b^2 + c^2 + \frac{2acy}{x} + \frac{a^2y^2}{x^2} = \frac{a^2y^2}{x^2} - \frac{2ay^2}{x} + y^2 + a^2 - 2ax + x^2$, ergo $\overline{b^2 + c^2} \times x - 2acy + 2ay^2 - y^2x = a^2x - 2ax^2 + x^3$. Ergo ſi $BF = BD$, & punctum F ſumatur principium abſciſſæ prodibit Æquatio formæ *Neutonianæ* $y^2x - 2acy = -x^3 + 4ax^2 + \overline{b^2 + c^2 - 5a^2} \times x + 2a^3 - 2b^2a - 2c^2a$.

Curva conſtabit ex Ovali *(Fig. 39.)* tranſeunte per C, D ac punctum E ſi BE = BC, atque ex Hyperbola Anguinea Aſſymptoton flexu contrario amplexa; nam Hyperbolam omnem non ad eaſdem partes poni Aſſymptoti hinc conſtat, quod ſi ad G punctum medium inter C & S, erigatur perpendicularis ipſi CS rectæ AR occurrens in T, & jungatur DT; ea producta Aſſymptoto occurret in Q, in eo puncto ubi curva Aſſymptoton ſecat; atque ea curvæ pars quæ verſatur inter Q & L eſt omnis inter rectas AR & FL; ea vero Curvæ pars quæ verſatur inter Q & H jacet omnis ad alias partes Aſſymptoti quam eſt AR. Quippe recta à C ducta ad punctum quodvis rectæ AR quod verſatur inter T & K, minor eſt recta à D ducta ad idem punctum; ſed RN = RD, & Rp = RC, adeoque Rp minor erit quam RN, adeoque puncta curvæ p verſantur inter AR & FL. Sed GT perpendicularis eſt in CD, & CG = GD, adeoque DT = CT, unde TQ = TD = Tp, & proinde curva Aſſymptoton ſecabit in Q. Quod ſi recta ducatur à C ad punctum rectæ AR inter T & O, ea erit major quam recta à D ad idem punctum rectæ AR ducta; unde cum Rp = RC, patet in hoc caſu p fore ad alias partes Aſſymptoti FH, & curvæ portio quæ verſatur inter Q & H jacebit extra Aſſymptoton. Hæc ex conſtructione manifeſta ſunt, & ex æquatione quoque radicum ope demonſtrari poſſent. Ex dictis patet curvam eſſe *ſpeciei* 33. *Neutonianarum* cui reſpondet *Fig.* 39. in Tractatu D. *Neutoni.* Notandum hujus curvæ proprietatem eſſe omnes rectas ad curvam terminatas concurrentes in D biſſecari recta AR.

Quod ſi C & D æqualiter diſtent à puncto B curva ſecabit Aſſymptoton in F; ſi vero C magis diſtet à B quam D, curva erit ejuſdem ſpeciei ſed ſecabit Aſſymptoton ad alias partes puncti F quam eſt Q. Notandum quoque eandem plane Curvam deſcribi ſi C ponatur ad alias partes rectæ AR in ſimili loco, *i. e.* ad eandem diſtantiam a BF & AR.

Corol. VIII.

Corol. VIII. Producatur DC (*Fig.* 40.) ad rectam AR; donec ei occurrat in X, & ſume XZ = XC, atque curva tranſibit per duo puncta C & Z. Unde ſi C ponatur in recta AR coincident C & R cum X, adeoque curva habebit punctum duplex in C, & Ovalis ac Anguinea jungentur ſeſe decuſſantes ad formam *Nodi* & curva evadet *ſpeciei* 34.

Corol. IX. Quod ſi punctum C (*Fig.* 41.) ponatur in recta DB, manifeſtum eſt curvam habere diametrum DB, quoniam ſi BR = Br, erit CR = Cr, RP = rp, adeoque DP = Dp, & PM = pM; quod inde quoque conſtat quod $c = o$, adeoque in æquatione evaneſcat terminus $2acy$. Curva vero in hoc caſu conſtabit ex Ovali tranſeunte per C & D, atque Conchoidali ad Aſſymptoton FH ſecante BF in S, ita ut BS = BC; quod ſi BC minor ſit DB, patet Ovalem eſſe ad Convexitatem Conchoidalis tranſeuntis per S, cum BS ſit minor BF; & proinde curva erit *ſpeciei* 39. At ſi BD minor ſit BC, erit BS major BF, adeoque Conchoidalis habebit Ovalem ad Concavitatem ſuam & curva erit *ſpeciei* 40.

Corol. X. Quod ſi punctum C coincidat cum B, coincidet etiam S cum B, adeoque erit B punctum curvæ duplex; Ovalis ac Conchoidalis *Corolarii* præcedentis ſe decuſſant in C, ad modum *Nodi* & curva erit *ſpeciei Quadrageſimæ primæ*; atque hæc curva eſt tertii Ordinis cujus Quadraturam exhibuit eximius Geometra D. de *Moivre* in Actis Philoſophicis, N° 345.

Corol. XI. Curvæ *Corol.* 8 & 10 (*Fig.* 42.) deſcriptæ, quæ ſcil. punctum habent duplex in C, ſic motu commodiſſime deſcribuntur. Sumatur angulus DPL = DCA, & ſit PQ æqualis datæ rectæ DC; moveatur angulus DPL ita ut punctum Q, quod datur in crure PL, deſcribat rectam AC & punctum angulare P deſcribet lineam vel *ſpeciei* 34 vel 41; ſi angulus DPL ſit obliquus erit curva *ſpeciei* 34, at ſi rectus curva erit *ſpeciei* 41. Quippe cum DPQ = DCR & CRD = PRQ, ſimilia erunt triangula PQR & DRC, ſed & æqualia ſunt cum DC = PQ; adeoque CR = RP; quæ ipſa eſt conſtructio quam exhibuimus *Corol.* 5. curvarum ſpecierum 34 & 41. Quod ſi punctum ſumatur medium inter P & Q Linea ejus motu deſcripta, ſi angulus DPQ rectus ſit, eſt *Ciſſois Dioclea*.

PROP.

Prop. XII.

Curvarum Prop. X. *deſcriptarum ſpecies & Aſſymptotos determinare.*

QUoniam Curvæ puncta determinantur concurſu rectarum *(Fig.* 28.*)* CO & SN, curva abibit in infinitum quoties rectæ CO & SN evadunt parallelæ; quod contingit ubi angulus CQL ſit æqualis ſummæ angulorum SNL & FCO.

Quare ſi in rectam NL *(Fig.* 43.*)* demittatur CQ, angulum CQL conſtituens æqualem ſummæ datorum SNL & FCO, & ad modum *Lemmatis* ſecundi curva hac ratione deſcribatur occurrens rectæ BQ in tribus punctis Q, *q*, *t*, jungantur CQ, C*q*, C*t*, & ducantur rectæ CO, C*o*, C*u* conſtituentes augulos QCO, *q*C*o*, *t*CU æquales dato FCO, erunt rectæ CO, C*o*, C*u* parallelæ Aſſymptotis curvarum *Prop.* 10. deſcriptarum.

Quod ſi ſumatur CH ad CS, ut angulus naſcens QC*m* ad NS*n*—QC*m*, & per H ducatur recta H*z* parallela rectæ CO, erit HZ curvæ Aſſymptotos & ſimili ratione duæ reliquæ determinantur.

Oſtendimus in *Lemmate* & *Corolariis* præcedentibus varias methodos, quibus curva hic ad Aſſymptoton poſitionem determinandam aſſumpta facile delineari poteſt, quas non opus eſt repetere: Ex ea vero deſcripta quot & qualia crura infinita habeat curva facile intelligitur. Quippe ſi BQ Hyperbolæ defectivæ ter occurrat curva habebit tres Aſſymptotos & ſex crura Hyperbolica; ſi BQ in puncto quovis Hyperbolam defectivam contingat Curvæ duæ Aſſymptoti abibunt in infinitum, & curva duo habebit crura Parabolica infinita, & duo Hyperbolica ſi BQ in alio puncto Hyperbolam defectivam ſecet. Nam ſi BQ Hyperbolam defectivam contingat in Q puncta quamproxima Q & *m* erunt ſimul in recta BQ & in Hyperbola illa, adeoque angulus C*m*L = CQL, & triangula CQ*g*, L*mg* erunt ſimilia; & proinde angulus QC*m* = NL*n* = NS*n*; cumque HC : CS :: QC*m*, NS*n* — QC*m*, patet HC evadere infinitam, adeoque Aſſymptoton HZ abire in infinitum; & proinde crura curvæ evadere Parabolica; cum vero in alio aliquo puncto recta BQ Hyperbolæ defectivæ occurrat, curva alia duo habebit crura Hyperbolica. Quod ſi Hyperbola defectiva habeat diametrum & ſit BQ ejus Aſſymptotos curva duo ſola habebit crura Parabolica infinita.

Ex

Ex *Corol.* 2. *Prop.* 10. intelligi poteſt an curva habeat *Nodum* in C *Cuſpidem* an *Punctum* conjugatum, unde ex dicto *Corolario* & ex hac Propoſitione collatis ſpecies curvæ dignoſci poteſt.

Corol. I. Si anguli SNL & FCO conficiant duos rectos, facilius determinantur Aſſymptoti & ſpecies curvarum; quippe imprimis manifeſtum eſt curvam abire in infinitum in hoc caſu quando Q abit ad diſtantiam infinitam; *(Fig.* 44.*)* quippe in eo caſu parallelæ evadunt NL & CF, adeoque cum SNL ſupplementum ſit anguli FCO ad duos rectos, parallelæ quoque evadunt CO & SN. Ducatur igitur CG conſtituens angulum GCS æqualem exceſſui anguli FCO ſupra CBQ, & CG erit parallela Aſſymptoto curvæ.

Corol. II. Porro ſuper *(Fig.* 45.*)* CS deſcribatur arcus circuli habens angulum SNL inſcriptum occurrens datæ AN in punctis duobus N & R; & ſi conſtituantur anguli NCP, RCπ æquales dato FCO, erunt CP & Cπ parallelæ aliis duabus curvæ Aſſymptotis; quarum Poſitio ſic inveſtigatur, ſit angulus SKN = SNL, & ſumatur NSU = KSN; ſitque CH ad CS ut eſt UQ ad CU, & per H ducatur HZ, ea erit Aſſymptos parallela ipſi CP, & ſimiliter determinatur Aſſymptotos quæ parallela eſt rectæ Cπ.

Corol. III. Duo vero ſunt caſus in quibus loco quatuor crurum Hyperbolicorum ad duas Aſſymptotos curva duo tantum habet crura Parabolica. 1. Quando BQ parallela eſt vel rectæ CN vel CR, nam in eo caſu erit UQ infinita, adeoque & CH cum CH : CS :: UQ : CU. Erit vero in hoc caſu curva Hyperbolo-Parabolica, cum alia habeat crura Hyperbolica ad Aſſymptoton parallelam ipſi Cπ. 2. Quando AN tangit circulum cui inſcribitur angulus SNL; quippe in eo caſu evadit angulus NSC = KSN, adeoque coincidit C cum U, & proinde cum CH : CS :: UQ : CU erit CH infinita, adeoque curvæ crura quæ prius erant ad Aſſymptotos parallelas rectis CP & Cπ evadunt Parabolica; alia vero manent in hoc caſu crura duo Hyperbolica ad Aſſymptoton parallelam ipſi CG.

Corol. IV. Quod ſi AN eum circulum contingat, & ſimul BQ parallela ſit rectæ CN, curva duo tantum habebit crura infinita eaque Parabolica; unde ſi angulus SCT æqualis ſit dato FCO & CT occurra rectæ BQ in T, atque per puncta C, S, T ducatur circulus; ſi is circulus bis occurrat rectæ AN curva habebit Ovalem & *Nodum* in S eritque Parabola *ſpeciei* 68; ſi AN circulum contingat curva erit *ſpeciei* 70. Parabola nimirum *Neiliana*, & denique ſi AN ponatur tota extra circulum curva erit Parabola *Punctata ſpeciei* 69.

Quoties recta AN cadit tota extra dictum circulum deſcriptum ſuper CS curva unicam habet Aſſymptoton parallelam rectæ CG, *(Fig.*44.*)* & curva evadit Hyperbola defectiva.

Ex his generalibus *Corolariis* innumeræ deduci poſſunt ſpeciales ſimplices curvarum Conſtructiones, quarum exemplum unum aut alterum afferre erit operæ pretium.

Ex. I. Sint anguli SNL, FCO recti *(Fig.* 46.) & AN perpendicularis in rectam CS atque BQ eidem CS parallela, dicatur CS $= a$, SA $= c$, AB $= b$, PM $= y$, & SM $= x$, eritque SM : PM :: SA : AN $= \frac{cy}{x}$ & CM $(= a - x)$: PM (y) :: QT $(= b)$: CT $= \frac{by}{a-x}$; ſed SM : PM :: BN : BQ, *i.e.* $x : y :: \frac{cy}{x} - b : \frac{by}{a-x} + a - c$, unde

$$cy^2x - cay^2 + baxy = \overline{a-c} \times x^3 - \overline{a-c} \times ax^2.$$

Proinde ſi ſumatur AD : AB :: $\frac{\text{CS}}{2}$: SA, & per D ducatur recta DH parallela rectæ CS; & dein ad C erigatur Normalis CK, occurrens rectæ DH in K; & K ſumatur pro principio abſciſſæ in axe DH prodibit æquatio curvæ formæ *Neutonianæ*

$$y^2x - \frac{ba^2}{c}y = \frac{a-c}{c} \times x^3 - \frac{2a-2c}{c} \times ax^2 + \overline{\frac{a-c}{c} + \frac{b^2}{4c^2}} \times a^2x - \frac{b^2a^3}{2c^2}.$$

Unde cum Terminus x^3 ſit affirmativus quoties SC ſuperat SA, & terminus y non deſit, patet curvam fore *ſpeciei* 7 vel 8; quippe ſi evaneſcat y & æquationis termini omnes ducantur in x & adjiciatur terminus $\frac{b^2a^4}{4c^2}$ prodibit æquatio cujus duæ radices erunt imaginariæ.

II. Quod ſi recta SA major ſit quam SC evadet terminus x^3 negativus, & ſi SA minor ſit quam $\frac{\sqrt{b^2+a^2}}{2} + \frac{a}{2}$ curva habebit *Nodum* in S, eritque *ſpeciei* 34; ſi SA $= \frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}$ curva erit *ſpeciei* 35, & ſi SA ſit major ea Quantitate curva erit *ſpeciei* 36.

III. Si recta AN abeat ad alias partes puncti S quam eſt C æquatio curvæ erit ſimilis priori, ſed termini x^3 coefficiens erit $\frac{-a-c}{c}$ adeoque negativus, & proinde curva erit Hyperbola defectiva. Unde ſi CA ſit minor quam $\frac{\sqrt{a^2+b^2}+a}{2}$ curva erit *ſpeciei* 34; ſi CA $= \frac{a+\sqrt{a^2+b^2}}{2}$ curva erit *ſpeciei* 35; ſi CA major ſit ea Quantitate curva erit *ſpeciei* 36; in primo caſu curva habet *Nodum* in S, in 2° *Cuſpidem*, in 3° *Punctum* conjugatum; & curvæ eæ omnes Aſſymptoton ſuam ſecant in puncto C.

IV. Sint rectæ BQ, AN *(Fig.* 47.) parallelæ rectæ CS, & anguli SNL, FCO ut prius recti; ducatur SA perpendicularis in AN occurrens rectæ BQ in B; & ſi CS major ſit quam 2 SA terminus x^3 in æquatione ad *Neutonianam* formam reducta erit affirmavimus, & curva erit Hyperbola redundans habens tres Aſſymptotos; ſi $2SA = CS$ evaneſcet terminus x^3, & curva evadet Hyperbolo-Parabolica; quod ſi 2SA major ſit quam CS terminus x^3 evadet negativus, & deſcribetur Hyperbola defectiva; hæc ſpecialius proſequi non opus eſt cum labore exiguo indagari poſſint.

V. Quæ hactenus deſcriptæ ſunt *(Fig.* 48.) curvæ diametris carent, aliæ vero quæ diametros habent deſcribi poſſunt, ſi loco anguli FCO ſubſtituatur recta circa C ut Polum volubilis; ſintque rectæ AN & BQ perpendiculares in CS. Quippe ſi $CB = b$, $SA = c$, $CS = a$, $PM = y$, $SM = x$, & CB minor ſit quam CA, erit $y^2x - \frac{ca}{c+b} \times y^2 = \frac{a-c-b}{c+b} \times x^3 - \frac{a-c-b}{c+b} \times ax^2$. Unde ſi ſumatur SD ad CS, ut ut eſt SA ad CB+SA, atque D ſumatur principium abſciſſæ, æquatio prodibit formæ *Neutonianæ* $y^2x = \frac{a-c-b}{c+b}x^3 + \frac{\overline{2ca-ba} \times \overline{a-c-b}}{\overline{c+b}^2}x^2 + \frac{\overline{c-2b} \times \overline{a-b-c}}{\overline{c+b}^3} \times a^2cx - \frac{bc^2a^3}{\overline{c+b}^3} \times \overline{a-c-b}$. Cujus æquationis Radices ſi evaneſcat y prodibunt $\frac{ca}{c+b}$, $\frac{ca}{c+b}$, $\frac{-ba}{c+b}$, unde manifeſtum eſt duas radices æquales, & tertiam eſſe ſigni contrarii. Idem hinc quoque conſtat quod DS, DS, DC ſint tres Radices æquationis cum curva tranſeat bis per punctum S & ſemel per C. Proinde cum terminus x^3 ſit affirmativus patet figuram eſſe *Cruciformem ſpeciei* 18 vel 19. Similiter ex aliis poſitionibus rectarum AN & BQ conſtruuntur *ſpecies* 11, 13, 41, &c.

Corol. V. Oſtendimus *Corol.* 2. *Prop.* 11. ſi SNL & FCO conſiciant duos rectos, & ſit angulus QBA = OCF deſcribi Conicam ſectionem cujus Aſſymptoti & crura ſic determinantur ex *Corol.* 2. bujus *Prop.* Super recta CS deſcribatur (*Fig.* 45.) circuli arcus habens angulum SNL ſibi inſcriptum; ſi hic circulus rectæ AN occurrat in duobus punctis N & R ducantur CN, CR & CP, Cπ angulos NCP, RCπ conſtituentes æquales dato FCO, & parallelæ erunt CP, Cπ Conicæ ſectionis Aſſymptotis quarum poſitio ſimiliter ac in *Corol.* 2. determinatur. Proinde ſi AN ſecet dictum circulum deſcribetur Hyperbola Conica, ſi circulum contingat Parabola, & ſi extra circulum tota ponatur deſcribetur Ellypſis.

Abſolvimus

Abſolvimus duas methodos quibus Lineæ tertii Ordinis puncto duplice donatæ ſola rectarum datarum & angulorum conſtantium ope deſcribuntur; quæ ſupereſt tertiam eas deſcribendi methodum breviter explicabimus; harum vero curvarum contemplationi prolixius immoramur propterea quod ſero in Geometriam receptæ ſint atque à paucis hucuſque tractatæ.

PROP. XIII.

Cæteris manentibus ut in Prop. 5. *ducatur crurum* (Fig. 49.) CF *&* NL *concurſus* Q *per rectam* BG *&* *crurum* CO *&* SN *concurſus* R *per* AR, *dico angulare punctum* N *deſcripturum Lineam tertii Ordinis.*

EX puncto C ad rectam BG ducatur CB conſtituens angulum CBG = FCO, & occurrat CB producta rectæ AR poſitione datæ in A; ducatur SA, & in recta SA ſumatur abſciſſa ut æquatio curvæ prodeat ſimplicior, quoniam ubi R pervenit ad A abeant Q & N in infinitum, & proinde SA fiat parallela Aſſymptotó curvæ; Ducatur NM ordinata ad Axem SA parallela rectæ BG & RK parallela quoque eidem rectæ BG occurrens axi SA in u, ſitque Qe parallela ipſi axi SA. His poſitis dicatur SA $= a$, CB $= b$, CA $= c$, AD $= e$, CD $= g$, ſitque Au : Ru :: a : d, &c. & quoniam SM ($= x$) : NM ($= y$) :: SU : Ru erit RU $= \frac{day}{dx - ay}$ & SU $= \frac{dax}{dx - ay}$, atque AU $= \frac{a^2y}{dx - ay}$. Sed UK:AU :: g : e ob parallelas RK, BG, CD & triangula proinde ſimilia AUK, ACD ergo UK $= \frac{a^2gy}{edx - eay}$ AK $= \frac{ca^2y}{edx - eay}$, unde RK $= \frac{\overline{ade + a^2g} \times y}{edx - eay}$, & CK $= \frac{ecdx - a \times \overline{ce + ca} \times y}{edx - eay}$. Ducatur CT conſtituens angulum CTB = CBT ſitque BT $= f$, & triangula CTQ, CRK erunt ſimilia; quippe CTQ = RKC & CQT = RCA cum utervis additus angulo QCB conficiat angulum DCB. Proinde eſt RK : CK :: CT : QT; unde

QT =

$QT = \frac{ebcdx - ba \times \overline{ce + ca} \times y}{a \times \overline{ed + ag} \times y}$, ſed $GQ = GB - BT - TQ$ & $GB = \frac{cg + gb}{c}$, $BT = f$, adeoque $GQ = \frac{cg + gb - fc}{c} - \frac{ebcdx + bc \times \overline{ec + ca} \times y}{\overline{ed + ag} \times ay}$ $= Me$, & $Qe = GM = \frac{\overline{c + b} \times e}{c} + a - x$.

Ducatur NP angulum NPS conſtituens æqualem dato SNL & ad eam rectam ducatur Sg conſtituens angulum $SgN = NMP$, ducatur quoque $Nm = NM$; ſintque rectæ NM, NP, PM, mP ut Quantitates datæ a, b, l, k reſpective; his poſitis cum ſimilia ſint triangula SgN & QeN erit $Sg : gN :: Ne : Qe$, ſed $Sg : SP :: a : b$ & $SP = SM + \frac{l \times NM}{a}$, ergo prodibit

$$\frac{ax + ly}{b} : \frac{by}{a} + \frac{kx}{b} + \frac{kly}{ab} :: y + \frac{cg + gb - fc}{c} - \frac{ebcdx - ba \times \overline{ce + ca} \times y}{\overline{ed + ga} \times ay} : \frac{\overline{c + b} \times e}{c}$$

$+ a - x$. Sed ſi centro N radio *(Fig.* 50.*)* NM (ubi NM evadit æqualis ipſi SA) deſcribatur circulus, is tranſibit per m & ſi producatur NP ad circulum occurret ei in punctis X & Y; ſed $MP : PX :: PY : Pm$ ex Elementis, adeoque cum $PX = a - b$, & $PY = a + b$, erit $kl = a^2 - b^2$, & $b^2 + kl = a^2$; quod ſi Analogiæ inveſtigatæ media & extrema in ſe mutuo ducantur, & loco $b^2 + kl$ ſubſtituatur a^2, æquatio prodibit hujuſmodi

$$a^2 \times \overline{de + ag} \times cy^3 + \overline{k + l} \times \overline{ed + ag} \times acxy^2 + a^2 \times \overline{ed + ag} \times cx^2y - bc^2dekx^2$$
$$\left.\begin{array}{l} + \overline{cg + gb - fc - lc} \times a^2 \times \overline{de + ag} \\ - ba^2c^2 \times \overline{e + a} - ale \times \overline{de + ag} \times \overline{c + b} \end{array}\right\} y^2 \left.\begin{array}{l} + \overline{cg + gb - fc} \times ak - a^2e \times \overline{c + b} \times \overline{de + ag} \\ - bc^2dae - a^3c \times \overline{de + ag} - bakc^2 \times \overline{e + a} \end{array}\right\} xy$$

$= 0$. Quæ eſt æquatio ad Lineam tertii Ordinis.

Corol. I. Erigatur SE perpendicularis *(Fig* 49.*)* ipſi SA, & quæratur æquatio ad axem parallelam ipſi SE, & per *Prop.* 8. prodibit formæ *Neutonianæ*, nam exterminabuntur xy^2, x^2 & xy; atque coefficiens termini x^3, poſtquam æquatio ad formam *Neutonianam* reducta eſt, erit (ſi b ſit ſinus anguli ADC ſumpto SA pro Radio) $\frac{2a \times \overline{k + l}^2 - 8a^3}{2ab^2} = \overline{k + l}^2 - 4a^2$

$= Mm^2 - \overline{NM + Nm}^2$, ſed duo latera NM & Nm, ſimul ſumpta majora ſunt tertio Mm, & proinde coefficiens termini x^3 erit negativus, adeoque curva deſcripta erit Hyperbola defecta ad unicam Aſſymptoton parallelam rectæ SA.

Corol. II. Si $y=0$ erit $bcdkx^2=0$, adeoque curva bis tranſibit per S quod punctum erit curvæ duplex; & ſi circa data *(Fig.* 51.*)* puncta C & S moveantur anguli dati QCR & QSR = QNR, ac concurſus crurum CR & SR ducatur per rectam AN, concurſus crurum CQ & SQ deſcribet conicam ſectionem occurrentem rectæ BG in duobus punctis D & F; cumque Q ad hæc puncta pervenerit in deſcriptione Propoſitionis curva tranſibit per S.

Corol. III. Hinc vero curvæ ſpecies dignoſcitur; quippe ſi recta BQ bis occurrat Conicæ ſectioni curva habebit *Nodum* in S, eritque *ſpeciei* 34 vel 41; ſi BQ ſectionem contingat curva erit *Cuſpidata* in S, adeoque *ſpeciei* 35 vel 42. Quod ſi ſectioni recta BQ non occurrat, nec eam contingat, curva habebit *Punctum* conjugatum in S, eritque *ſpeciei* vel 36, 43 vel 44.

Corol. IV. Quod ſi angulus datus SNL æqualis ſit angulo AGB, erit NPM = PMN = NnM, & proinde coincident P & *m*, & quantitas *k* evaneſcet, adeoque terminus $bcdkx^2$, unde omnes æquationis termini dividi poterint per y; & proinde curva erit ſectio Conica.

Corol. V. Si anguli QSR & QCR ſint ejuſmodi *(Fig.* 51.*)* ut SQ & CQ ſimul coincidant cum CS punctum Q deſcribet Lineam rectam, ut conſtat ex *Prop.* 3. adeoque punctum S evadet ſimplex, & curva erit ſectio Conica.

COROL. GENERALE.

Ex Propoſitionibus 5, 10, 13 *univerſaliter colligi poteſt, quod ſi in plano dentur duo puncta* (Fig. 52.) C *&* S, *& ex* S *exeat recta* SN *cui ſemper occurrat ad angulum datum recta* NL *ad punctum* N, *circa* C *moveatur ut Polum angulus datus* FCO, *& crurum* CF *&* NL *concurſus ſit* P, *concurſus crurum* SN *&* CO *ſit* Q, *atque trium punctorum* Q, P, N *(in Quadrilaturo* CQNP*) duo quævis per rectas poſitione datas ducantur, tertium deſcribet Lineam tertii Ordinis habentem punctum duplex vel in* C *vel in* S.

SECTIO

SECTIO III.

De deſcriptione Linearum quarti Ordinis, earumque tertii Ordinis quæ nullum habent Punctum duplex.

Lineas ſecundi Ordinis deſcripſimus *Prop.* 1. motu duorum angulorum circa data duo puncta; Lineas tertii Ordinis deſcripſimus *Prop.* 5. & 10. ope totidem angulorum quorum alterum circa datum punctum moveri, alterum vero per rectam datam duci ſuppoſuimus; atque nunc ad Lineas quarti Ordinis deſcribendas uterque angulus per rectam datam eſt ducendus.

PROP. XIV.

Angulorum datorum (Fig. 53.) CNL, SRQ *crura* CN & SR *ſemper tranſeant per data puncta* C & S; *Puncta vero angularia* N & R *percurrant Rectas infinitas* AN & DR; *dico ſi concurſus crurum* NL & RQ *ducatur per Rectam* BQ *concurſum crurum* CN & SR *deſcripturum Lineam quarti Ordinis.*

Sint rectæ PM, NU, QK & RT normales in CS; ſit angulus CON=CNL, & SHR=SRQ; dicatur CS=a, CA=c, CB=b, SD=d; ſit $Au : Nu :: a : e$, $DT : RT :: a : g$, ſinus anguli CNQ ad coſinum ut a ad f, & ſinus anguli SRQ ad ſinum ejus complementi ut a ad k, & $BK : QK :: a : b$, PM=y, CM=x, SM =$a-x$.

$= a - x$. His poſitis eſt CM : PM :: CU : NU; adeoque $NU = \frac{ecy}{ex - ay}$ & $CU = \frac{ecx}{ex - ay}$. In N*o* cadat C*g* normalis, & ſimilia erunt triangula C*g*N & N*e*Q, atque C*og* & N*g*F; unde C*o* : NF :: N*e* : Q*e*; ſed $CO = CU - \frac{fNU}{a}$, & $NF = NU + f\frac{CU}{a}$. Quare ſi $QK = z$, erit $x - \frac{fy}{a} : y + \frac{fx}{a} :: \frac{ecy}{ex - ay} - z : b + \frac{az}{b} - \frac{ecx}{ex - ay}$, unde

$$QK\ (= z) = \frac{\overline{c - b} \times abex^2 + \overline{ce - bf} \times aby^2 + \overline{ba^2 + efb} \times bxy}{\overline{a^2 + bf} \times ex^2 - \overline{a^2 - bf + eb - ef} \times dxy - a^2 \times \overline{b - f} \times y^2}$$

Simili plane ratione alius inveſtigatur valor rectæ QK ex conſideratione Poſitionis rectæ DR & anguli SRQ; qui facile prodibit ſi loco ipſius x ponatur $a - x$, pro c ſcribatur d, atque pro f & e ſcribantur k & g, & denique pro $b + \frac{az}{b}$ ſcribatur $a - b - \frac{az}{b}$ in Analogia præcedenti. Duo vero hi valores æquentur &

$$\frac{\overline{c - b} \times aebx^2 + \overline{ce - bf} \times bay^2 + \overline{ba^2 + efb} \times bxy}{\overline{a^2 + bf} \times ex^2 - \overline{a^2 - bf + eb - ef} \times axy - a^2 \times \overline{b - f} \times y^2} =$$

$$\frac{\overline{a - b - d} \times gab \times \overline{a - x}^2 - \overline{a - b} \times \overline{a^2 + kg} \times \overline{a - x} \times by + \overline{a - b} \times \overline{k - dg} \times aby^2}{\overline{a^2 + kb} \times g \times \overline{a - x}^2 - \overline{a^2 - bk + gb - gk} \times \overline{a - x} \times y - a^2 \times \overline{b - k} \times y^2}$$

Unde ducendo numeratorem unius fractionis in denominatorem alterius prodibit æquatio $Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4 + Fx^3 + Gx^2y + Hxy^2 + Iy^3 + Kx^2 + Lxy + My^2 = o$, cujus coefficientes calculi ope indagari poſſunt, & ſpeciatim $\frac{A}{a^2b^2e} = \frac{a}{b} + \frac{k}{a} \times \overline{c - b - a - b - d} \times \frac{a}{b} + \frac{f}{a}$ & $F = -2aA$, $K = a^2A$, ut cuivis vel leviter perpendenti manifeſtum erit. Cum igitur x & y ad quatuor aſcendant Dimenſiones in ea æquatione & non ad plures, Linea erit Ordinis quarti.

Corol. I. Curva habebit duo puncta duplicia alterum in C, alterum in S; nam ſi $y = o$, evaneſcent omnes æquationis termini præter tres in quibus non reperitur y, & fiet

$$\overline{a^2 + kb} \times \overline{c - b - a - b - d} \times \overline{a^2 + fb} \times \overline{x^4 - 2ax^3 + a^2x^2} = o,$$

cujus radices ſunt o, o, a, a. Unde curva bis tranſibit per principium abſciſſæ quod eſt in C, & bis quoque tranſibit per S, cum $CS = a$.

Corol. II. Curva tranſit per C (*Fig.* 54.) quando recta SR pervenit ad Poſitionem SD; unde ſi angulus SDE = SRQ, & ſi ſuper CE deſcribatur arcus circuli cui inſcriptus ſit angulus CNL, atque is circulus bis

occurrat

occurrat rectæ AN, curva habebit *Nodum* in C cum ovali vel *Crucem*; at ſi recta AN eum circulum contingat curva erit *Cuſpidata* in C, & ſi AN ponatur tota extra circulum curva habebit *Punctum* conjugatum in C abſque *Nodo*, ovali aut *Cuſpide*.

Corol. III. Similiter plane ſi angulus CAG = CNL, & circulus ſuper GS deſcriptus habens angulum æqualem dato SRQ inſcriptum bis occurrat rectæ DR, curva erit *Nodata* in S; ſi rectam tangat circulus curva erit *Cuſpidata* in S, at ſi cadat DR penitus extra circulum dictum curva erit *Punctata* in S.

Corol. IV. Hinc curvæ hac Propoſitione deſcriptæ in ſex inſigniores ſpecies diſtinguuntur, quæ in inferiores deinceps dividi poſſunt. 1ª. Eſt earum quæ duos habent *Nodos*, alterum in C, alterum in S. 2. Eſt earum quæ duas habent *Cuſpides* in iiſdem punctis. 3. Earum quæ duo habent *Puncta* conjugata, alterum in C, alterum in S. 4. Earum quæ habent unum *Nodum* & unicam *Cuſpidem*. 5. Earum quæ habent unicum *Nodum* & unicum *Punctum* conjugatum. 6. Eſt earum quæ unicam habent *Cuſpidem* & unicum *Punctum* conjugatum.

Corol. V. Quod ſi, cæteris (*Fig.* 53.) manentibus ut prius, curva non determinetur concurſu rectarum CN & SR, ſed aliarum rectarum quæ cum CN & SR datos conſtituunt angulos ad puncta C & S, curva tamen non alia deſcribetur quam quæ methodo hujus *Prop.* deſcribi poteſt, ut ex *Lemmate* 1. facile demonſtratur.

Corol. VI. Curvæ hujus *Prop.* octo habere poſſunt crura infinita ad quatuor Aſſymptotos; cum vero eæ aliter determinari non poſſint quam concurſu rectæ cum linea curva Ordinis quarti, iis determinandis non immorabimur. Sed ſi anguli SNL, SRQ ſimul conficiant duos rectos tres Aſſymptoti curvarum determinari poſſunt deſcribendo Hyperbolam defectivam ad modum *Lemmatis* 2. dimittendo ſcil. ex C (*Fig.* 53) in RQ lineam CX, ita ut angulus CXR ſemper æqualis ſit dato CNL, dum punctum R ducitur per rectam DR; nam ſi à puncto C ad hujus Hyperbolæ occurſus cum AN ducantur rectæ, eæ erunt curvæ Aſſymptotis parallelæ.

PROP. XV.

Si rectæ SP *&* CP *ſimul coincidant cum* CS, *curva* Prop. 14. *deſcripta non erit quarti ſed tertii Ordinis, & puncta* C *ac* S *erunt ſimplicia.*

COincidat CN cum CA, & NL cum AG (*Fig.* 55.) atque ſimul coincidat SR cum SD & RQ cum DG, dico Lineam tertii Ordinis in eo caſu methodo *Prop.* 14. delineari; nam erit BO : GO :: $a : b$ & AO : GO :: $f : a$; DO : GO :: $k : a$, adeoque $\frac{a}{b} + \frac{f}{a} = \frac{BO}{GO} + \frac{AO}{GO} = \frac{AB}{GO}$, atque $\frac{a}{b} + \frac{k}{a} = \frac{BO}{GO} + \frac{DO}{GO} = \frac{BD}{GO}$, unde $\overline{\frac{a}{b} + \frac{f}{a}} : \overline{\frac{a}{b} + \frac{k}{a}}$ :: AB : BD :: $c - b : a - b - d$, & proinde $\overline{a - b - d} \times \overline{\frac{a}{b} + \frac{f}{a}} = \overline{c - b} \times \overline{\frac{a}{b} + \frac{k}{a}}$; unde termini æquationis Propoſitione præcedente inveſtigatæ in quibus y non reperitur $\overline{a^2 + kb} \times \overline{c - b} - \overline{a - b - d} \times \overline{a^2 + fb} \times \overline{x^4 - 2ax^3 + a^2x^2}$ evaneſcunt: Reliqui dividantur per y, & æquatio evadet dimenſionis unitate inferioris quam prius, adeoque curva erit Linea Ordinis tertii, cujus termini erunt x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y cum ſuis coefficientibus; cum vero in ea æquatione reperiantur y & x patet principium abſciſſæ quod eſt in C eſſe punctum ſimplex; proinde C & S erunt puncta duo ſimplicia.

Corol. I. Producatur GD & ſumatur angulus DSE = FDE & PCA = QAG; ſintque rectæ PX & ET perpendiculares in rectam BG, & ſumatur CL ad SL ut ET × PG² ad PX × EG², eritque L punctum tertium præter C & S ubi curva deſcripta ſecat rectam CS. Hæc innoteſcunt inveſtigando concurſum rectarum CN & SR cum quamproxime coincidunt cum CS.

Corol. II. Ex Propoſitione conſtat ſi rectæ AN & DR in eodem puncto ſecant rectam CS & angulus CNL ſupplementum ſit anguli SRQ ad duos rectos deſcribi Lineam tertii Ordinis; nam ubi ſimul coincidunt anguli CNL & SRQ cum CAG & SDG, & punctum A coincidit

cidit cum D, patet angulos CAG & SDG evadere ſupplementa ſibi mutuo ad duos rectos.

Corol. III. Si anguli CNL & SRQ (*Fig.* 56.) ſimul ſumpti conficiant duos rectos, & CBQ = CNL, Linea deſcribetur tertii Ordinis; quippe cum N pervenerit ad A abibit Q in infinitum, cum CNL = CBQ, ſed SRQ = SBQ, adeoque SR ſimul coincidet cum SD, unde per *Prop.* Linea deſcribetur tertii Ordinis.

Corol. IV. Unde ſi anguli CNL & SRQ ſint recti atque vel coeant puncta A & D ut in *Corolario* 2 vel recta BQ perpendicularis ſit in CS ſecundum *Corol.* 3. linea deſcripta erit tertii Ordinis.

Corol. V. Sit imprimis BQ (*Fig.* 57.) perpendicularis in CS & æquatio curvæ erit hujuſmodi iiſdem retentis ſymbolis

$$\overline{b-c}\times x^2 - cy^2 + \frac{ba}{e}xy\times\overline{a-x-\frac{ay}{g}} = \overline{a-d-b}\times\overline{a-x}^2 - dy^2 + \frac{ba}{g}y\times\overline{a-x}\times\overline{x-\frac{ay}{e}}$$

unde ſi rectæ AN & DR ſint quoque perpendiculares in CS, & proinde parallelæ rectæ BQ æquatio erit hujus formæ

$$y^2x - \frac{cay^2}{c+d} = \frac{a-d-c}{d+c}\times x^3 - \frac{2a-2d-b-c}{d+c}\times x^2 + \frac{a-b-d}{d+c}\times a^2x$$

ſumatur CF ad SF ut CA ad SD ſeu CF ad CS ut CA ad CA+SD; & ſi principium abſciſſæ ſumatur in F æquatio curvæ erit formæ *Neutonianæ*,

$$y^2x = \frac{a-d-c}{d+c}\times x^3 + \frac{3c\times\overline{a-d-c}}{\overline{c+d}^2} + \frac{b+c+2d-2a}{d+c}\times ax^2$$

$$+\frac{a-b-d}{d+c}\times a^2 + \frac{3c^2a^2\times\overline{a-c-d}}{\overline{c+d}^3} + \frac{2a^2c}{\overline{c+d}^2}\times\overline{b+c+2d-2a}\times x$$

$$+\frac{a^3cd}{\overline{d+c}^3}\times\overline{a-b-d} - \frac{a^3c^2d}{\overline{d+c}^4}\times\overline{a-d-c} = 0.$$

Sit $y = 0$ & æquationis Radices erunt $\frac{ca}{d+c}$, $\frac{ca}{d+c} - \frac{a\times\overline{a-b-d}}{a-d-c}$, $-\frac{da}{d+c}$; quod abſque prolixo quovis calculo cuivis manifeſtum erit, qui harum ſummam cum æquationis termini ſecundi coefficiente, & productum ex earum multiplicatione reſultantem cum ipſo termino ultimo contulerit, poſtquam omnes termini per coefficientem termini primi dividuntur. Cum vero eæ radices ſint omnes inæquales & figura diametrum habeat ob defectum termini y patet Lineam nullum habere punctum duplex.

Ejuſdem æquationis tres radices (*Fig.* 58.) ſic facile geometrice determinantur. In recta CS ſumatur imprimis punctum F ita ut CF : SF :: CA : SD. Dein ſumatur punctum L ita ut CL : SL :: DB : AB; erunt rectæ FC, FS, FL tres radices æquationis & C, S, L tria deſignant

nant puncta ubi curva occurrit rectæ CS. Punctum vero F quod principium eſt abſciſſæ concurſus eſt rectæ CS cum Aſſymptoto curvæ ei perpendiculari.

Corol. VI. Hinc ſi AC ſit minor quam DC erit $a - d - c$ quantitas affirmativa, adeoque terminus $\frac{a - d - c}{d + c} x^3$ erit affirmativus; cumque F ſit punctum inter C & S adeoque ſigna radicum FC & FS ſint diverſa, patet Lineam in hoc caſu deſcribi tertii Ordinis ſpeciei vel 20 vel 21, & proinde figuram conſtare ex duabus Hyperbolis in oppoſitis angulis duarum Aſſymptotωn poſitis, & conchoidali intermedia ad Aſſymptoton perpendiculariter inſiſtentem rectæ CS in puncto F; quoties reliquæ duæ Aſſymptoti concurrunt ad eaſdem partes puncti F ad quas eſt L curva erit ſpeciei 20, eritque 21 quoties concurrunt ad diverſas. Aſſymptoti curvarum hac *Prop.* deſcriptarum (quoties CNL & SRQ conficiunt ſimul duos rectos), determinantur (*Fig.* 59.) ope Hyperbolæ defectivæ deſcriptæ ſecundum *Lemma* 2, faciendo ut rectus angulus SRQ angulari ſuo puncto R percurrat rectam datam DR, & ex C demittendo perpendicularem CH. Quippe ſi à puncto C ad puncta ubi Hyperbola hæc, quam H perpetuo tangit, occurrit rectæ AN ducantur lineæ rectæ, eæ erunt parallelæ Aſſymptotis curvarum hac *Prop.* deſcriptarum. Sed ex *Lemmate* 2° manifeſtum eſt Hyperbolam ad Aſſymptotos determinandas aſſumptam tranſire per D & C, atque unicam ejus Aſſymptoton tantum diſtare à C quantum D ab S; unde in hoc caſu ubi D eſt inter C & S patet Hyperbolam ovalem habere per C & D tranſeuntem, quæ bis occurret rectæ AN, cum CA minor ſit CD; adeoque curva habebit duas Aſſymptotos præter eam quæ perpendicularis eſt rectæ CS in F, quæ formatur cum recta AN occurrit Hyperbolæ ad diſtantiam infinitam. Imprimis igitur oſtendimus tum ex curvæ æquatione tum ex *Lemmate* 2. & *Corol.* 6. *Prop.* 14. curvam habere tres Aſſymptotos quoties D eſt inter C & S atque A inter C & D.

Corol. VII. Sit nunc CA (*Fig.* 60.) major quam CD manente puncto B, & $d + c$ major erit quantitas quam a, & proindè $a - d - c$ erit quantitas negativa, adeoque terminus $\frac{a - d - c}{d + c} x^3$ negativus; cumque radices CF & SF ſint adhuc ſigni diverſi, & curva diametrum habeat, patet Lineam tertii Ordinis deſcribi *ſpeciei quadrageſimæ*, & figuram conſtare ex Conchoidali ad Aſſymptoton FH, & Ovali tranſeunte vel per puncta C & L vel per S & L. Hæc ex *Lemmate* 2° quoque facile conſtant cum Hyperbola defectiva ad Aſſymptotos determinandas aſſumpta tranſeat per C & D, ejuſque Aſſymptotos ſit KG normalis in CK, ſi CK = SD. Nam in hoc caſu cum CA ſit major CD, recta AN Hyperbolæ non poteſt occurrere niſi ad diſtantiam infinitam, adeoque

Prop. XVII.

Moveantur ut in Prop. 14. (Fig. 64.) *anguli dati* CNL & SRQ; *ſed nunc concurſus crurum* CN & SR *ducatur per rectam infinitam* BT, *dico crurum* NL & QR *concurſum* P *deſcripturum Lineam quarti Ordinis.*

Sint rectæ NU, PM, RX, TK perpendiculares in CS; ſit quoque angulus NOC = CNL, & SHR = SRQ, ſit Cg normalis ex C in NO & SI normalis in RH, ſit CS = a, CB = b, CA = c, SD = d, PM = y, CM = x. AU : NU :: $a : e$; BK : KT :: $a : b$; DX : XR :: $a : g$; ſitque ut prius ſinus anguli SRQ ad ejus coſinum ut a ad k, & ſinus anguli CNL ad coſinum ut a ad f.

His poſitis cum triangula CgN & NPe atque triangula COg & NgF ſint ſimilia, erit CO : NF :: Ne : Pe. Sed CO = CU $-\frac{f\text{NU}}{a}$ & NF = NU $+\frac{f\text{CU}}{a}$. Unde ſi dicatur ordinata NU = z erit

$$e+\frac{az}{e}-\frac{fz}{a} : z+\frac{fc}{a}+\frac{f}{e}z :: z-y : x-c-\frac{az}{e},$$ unde $$\frac{e^2+a^2}{e^2}\times z^2 - \overline{\frac{e+f}{e}}\times y + \overline{\frac{ef-a^2}{ea}}\times x - \overline{\frac{2ac}{e}}\times z = cx-c^2+\frac{fcy}{a},$$ ſed CU :: NU :: CK : TK. Unde ſi TK = u erit $z=\frac{ebcu}{au\times\overline{e-b}+beb}$, & proinde

$$\left.\begin{matrix} ac^2b^2\times\overline{a^2+e^2}+a^2\times\overline{e^2-b^2}\times\overline{cx-c^2} \\ \overline{ef-a^2}\times x-\overline{e+f}\times ay\times cab\times\overline{e-b}+e^2b^2fcy \end{matrix}\right\} u^2 \left.\begin{matrix} \overline{ef-3a^2}\times x-\overline{e+3f}\times ay\times bceb^2 \\ +2babe^2\times\overline{acx-ac^2+fcy} \end{matrix}\right\} u$$
$$=b^2ce^2b^2ax-b^2c^2e^2b^2a+b^2fce^2b^2y.$$

Similiter ex conſideratione rectæ DR, & anguli SRQ, alius inveſtigari poteſt valor rectæ KT (= u) quæ ex hac facile deduci poteſt ſubſtituendo loco e, f, c, x, b ſymbolos g, k, d, $a-x$, $a-b$ cum eadem plane ratione inveſtigandus ſit valor KT ac prius niſi quod pro rectis AN, CM, CB & angulo CNL & nunc ſubſtituendæ ſint DR, SM, SB & angu-

& angulus SRQ; Sit m coefficiens termini u^2 in prima æquatione, & p coefficiens termini u in eadem, & ſi in iis coefficientibus loco e, f, c, x, b ſcribantur g, k, d, $a-x$, $a-b$ prodibunt duæ coefficientes terminorum u^2, u in ſecunda æquatione; quæ dicantur n & q; proinde ſi dicatur $b^2cx - b^2c^2 + \frac{fcb^2y}{a} = \frac{r}{e^2b^2a}$ & $\overline{a-b}^2 \times d \times \overline{a-x} - \overline{a-b}^2 \times d^2 + \overline{a-b}^2 \times \frac{kdy}{a} = \frac{l}{g^2b^2a}$ duo valores rectæ CT his æquationibus exprimi poterint quarum coefficientes ex dictis colligi poſſunt $\begin{matrix} mu^2 + pu = r \\ nu^2 + qu = l \end{matrix}$

Ex ſecunda patet $u^2 = \frac{l-qu}{n}$ & ex prima $u^2 = \frac{r-pu}{m}$, unde $\frac{l-qu}{b} = \frac{r-pu}{m}$ & $u = \frac{rn-lm}{pn-qm}$. In æquationum prima ſcribe loco u hunc ejus valorem & prodibit $\frac{mr^2n^2 - 2rnlm^2 + l^2m^3 + p \times \overline{rn-lm} \times \overline{pn-qm}}{\overline{pn-qm}^2} = r$ unde reducendo prodit $r^2n^2 - 2rnlm + l^2m^2 - p^2nl + pqml + pqnr - q^2mr = 0$.

Cum vero in terminorum r, n, l, m, p, q valoribus quantitates x & y ad unam tantum aſcendant dimenſionem, & in terminis æquationis $r^2n^2 - 2rnlm + l^2m^2 - p^2nl + pqml + pqnd - q^2mr = 0$ quatuor ſint factores, patet ſi ſubſtituantur loco r, n, l, m, p, q eorum valores æquationem prodituram in qua x & y erunt quatuor dimenſionum; & proinde lineam deſcribi quarti Ordinis.

Corol. I. Curva tranſibit neque per punctum C neque per S; quippe punctum curvæ ſemper verſatur in recta NL, quæ nunquam poteſt tranſire per C, cum rectæ NL & NC in unico tantum puncto N poſſint ſibi mutuo occurrere, & eodem argumento conſtat curvam non poſſe tranſire per S.

Corol. II. Super recta CS (*Fig.* 65.) deſcribatur circulus, habens angulum inſcriptum æqualem ſummæ datorum CNL & SRQ, occurrens rectæ BT in punctis duobus H & K. Ducantur CH, CK & rectæ HO & KO ita ut anguli OHC, OKC æquales ſint dato CNL; atque rectæ HO & KO erunt parallelæ duabus curvæ Aſſymptotis. Curva vero alias duas Aſſymptotos poteſt habere quarum poſitio ſic determinatur; ſit C△ parallela ipſi AN occurrens rectæ BT in △; jungatur S△ occurrens rectæ DR in R; atque ſit SRZ æqualis dato SRQ in *Fig.* 64. & recta RZ oſtendet plagam aliorum crurum infinitorum, & ſimiliter quartæ Aſſymptoti directio inveniri poteſt. Curvæ vero hæ ſemper

per habent quatuor crura infinita, adeoque eæ quarti Ordinis, quæ nulla habent crura infinita ſed in ſe redeunt, hac ratione deſcribi non poſſunt.

Prop. XVIII.

Si in Prop. *præcedente ſint* AN, BT, DR *parallelæ, Linea deſcripta erit tertii Ordinis.*

Quippe in hoc caſu manentibus iiſdem ſymbolis erit $z = \frac{cu}{b}$ atque ideo $\frac{\overline{e^2 + a^2}}{e^2 b^2} c^2 u^2 +$ &c. $= cx - c^2 + \frac{fcy}{a}$, & $\frac{\overline{g^2 + a^2}}{g^2 \times \overline{a - b}^2} \times d^2 u^2 +$ &c: $= d \times \overline{a - x - d}^2 + \frac{kdy}{a}$, unde $m = \frac{\overline{e^2 + a^2} \times c^2}{e^2 b^2}$, & $n = \frac{\overline{g^2 + a^2} \times d^2}{g^2 \times \overline{a - b}^2}$ adeoque m & n evadunt quantitates invariatæ; ſed $r^2 n^2 - 2nlm + l^2 m^2 - p^2 nl + pqml + pqnr - q^2 mr = 0$, adeoque cum m & n ſint quantitates datæ, ſi loco p, q, l, r earum valores ſubſtituantur patet in æquatione reſultante x & y ad tres tantum dimenſiones aſcendere, adeoque Lineam deſcribi Ordinis tertii.

Corol. I. Ex conſtructione *Prop.* 17. patet univerſaliter ſi rectæ AN, BT, DR concurrant in uno quovis puncto curvam per id punctum tranſituram. Cum igitur in caſu hujus *Prop.* hæ rectæ concurrant ad diſtantiam infinitam patet curvæ Aſſymptoton aliquam rectis his parallelam eſſe; reliquæ duæ ope circuli determinantur ut in *Corol.* 2. *Prop.* præcedentis.

Corol. II. Cum harum curvarum Aſſymptoti adeo facile determinentur, eæ ad *Neutonianas* minori labore reduci poterint inveſtigando æquationem curvæ ad aliquam Aſſymptoton ut exterminetur terminus y^3, & dein ad modum *Prop.* 8. exterminando terminum $x^2 y$. Quod ſi anguli CNL & SRQ conficiant duos rectos curvæ erunt Hyperbolo-Parabolicæ duo habentes crura infinita Parabolica, & duo Hyperbolica ad Aſſymptoton quæ cum CS angulum conſtituit æqualem ipſi CNL & rectæ CS occurrit in puncto O ita poſito ut AO : AD :: SB : CS.

Corol. III.

Corol. III. Sint e. g. anguli *(Fig. 67.)* CNL, SRQ recti & Lineæ AN, BT, DR omnes perpendiculares in CS, eritque ſi DR $= r$, NA $= z$, $r^2 - ry = d \times \overline{a - d - x}$, & $z^2 - zy = cx - c^2$, ſed $c : b :: z :$ BT $= \frac{bz}{c}$ & $d : r :: a - b :$ BT $= \frac{\overline{a - b} \times r}{d}$, unde $r = \frac{dbz}{\overline{a - b} \times c}$ atque adeo erit $z^2 - zy = cx - c^2$ & $\frac{d^2 b^2}{\overline{a - b}^2 \times c^2} \times z^2 - \frac{dby}{\overline{a - b} \times c} z = d \times \overline{a - d - x}$ quare cum $m = 1$, $n = \frac{d^2 b^2}{\overline{a - b}^2 \times c^2}$, $p = -y$, $q = -\frac{dby}{\overline{a - b} \times c}$, $r = cx - c^2$, $l = d \times \overline{a - d - x}$; Prodibit loco ſymbolorum m, n, p, q, r, l, ſubſtituendo eorum valores æquatio hujus formæ $Ay^2x + By^2 = Cx^2 + Dx + E$ cujus coefficientes ſimplici calculo ſed prolixo ſunt inveſtigandæ; dividantur omnes termini per A coefficientem termini y^2x; atque ſi CF ſumatur æqualis $\frac{B}{A}$ coefficienti termini y^2, & F ſit principium abſciſſæ, æquatio prodibit formæ *Neutonianæ* $y^2x = bx^2 + cx + d$, ubi cum deſit terminus x^3 patet figuram eſſe Hyperbolo-Parabolicam, & diametrum habere CS quoniam deeſt terminus y.

Corol. V. Sint anguli CNL, SRQ ut prius recti, *(Fig.68.)* ſed nunc ſint Lineæ omnes AN, BT, DR parallelæ rectæ CS; ſitque CA $= c$, CB $= b$, CD $= d$, CU $= z$, & SX $= r$, & patebit eſſe $z^2 - xz = cy - c^2$, & $r^2 - \overline{a - x} \times r = dy - d^2$ atque $dac - bdz = bcr$, unde erit $z^2 - \frac{2ac}{b} + \frac{\overline{a - x} \times c}{d} \times z = \frac{\overline{y - d} \times d^2}{d} - \frac{a^2 c^2}{b^2} + \frac{ac^2 \times \overline{a - x}}{bd}$, unde cum $m = n = 1$ & $p = -x$; $q = \frac{\overline{b - 2d} \times ac - bcx}{bd}$, $r = c \times \overline{y - c}$ & $l = \overline{y - d} \times d + \frac{\overline{b - d}\, a^2c^2 - abc^2x}{b^2 d}$ ſi ſubſtituantur loco p, q, r, l, hi earum valores prodibit æquatio ubi deerunt termini y^3, x^2y, xy^2 quæ proinde ad caſum 3. D. *Neutoni* reduci poteſt $y^2 = ax^3 + bx^2 + cx + d$. Curvam hoc *Corol.* deſcriptam Parabolam eſſe ex *Corol.* 2. quoque conſtat; nam ibi AO : SB :: AD : CS, adeoque cum puncta A & D in infinitum abeant, patet O quoque in infinitum abire, adeoque curvam reddi pure Parabolicam. Harum æquationes earumque radicum inveſtigatio adeo prolixum poſtulant calculum ut hic commode deſcribi non poſſint. Qui

hiſce delectantur modum caculi ex dictis facile percipient, & majore cum voluptate calculum ipſi repetere poterunt quam noſtrum perlegerent.

Corol. VI. Si p & q evadunt quoque invariatæ curva erit ſectio Conica, nam cum m & n ſint invariatæ ſoli quatuor ultimi termini æquationis

$$r^2n^2 - 2rnlm + l^2m^2 - p^2nl + pqml + pqnr - q^2mr = 0,$$

eam reddunt trium dimenſionum, in quibus reperitur vel p vel q, unde ſi p & q ſint invariati & loco m, n, p, q, l, r ſubſtituantur earum valores æquatio reſultans erit ad ſectionem Conicam.

Corol. VII. Similiter demonſtrari poteſt, quod ſi in conſtructione *Prop.* 17. coincidat recta AN cum DR & anguli CNL, SRQ ſint recti, Lineam deſcribi tertii Ordinis.

Prop. XIX.

Cæteris manentibus ut in Prop. 14. *concurſus crurum* CN & SR (Fig, 69.) *ducatur per rectam infinitam* BT; *dico anguli* CNL *punctum angulare* N *deſcripturum Lineam quarti Ordinis.*

SIT NM normalis in CS æqualis y, CM $= x$, CA $= a$, CB $= b$, SD $= d$, &c. & imprimis quæratur valor rectæ QP dicendo (ſi QP $= z$) ut $ax - fy : ay + fx :: ey - ez : ec + az - ex$. Unde $z = \dfrac{aey^2 + aex^2 + \overline{fy - ax} \times ce}{ef + a^2 \times x + \overline{e - f} \times ay}$. Porro CM : NM :: CU : UT & proinde erit TU $= \dfrac{bby}{bx - ay}$ & SU $= \dfrac{\overline{a-b} \times \overline{bx - a^2y}}{bx - ag}$, ſed SU : TU :: SK : RK, unde (ſi RK $= z$) erit $\overline{a-b} \times \overline{bx - a^2y} : bby :: gd + az : gz$ & proinde RK $= \dfrac{gbdy}{g \times \overline{a-b} \times bx - a^2gy - baby}$. Sed SK $- \dfrac{RKf}{a}$: RK $+ \dfrac{SKf}{a}$ :: RK $-$ QP : PK, unde

$$\overline{a-b} \times adxb - a^3dy - dbbfy : adbby + \overline{a-b} \times \overline{dfbx - a^2dfy}$$

$$:: \frac{dbgby}{\overline{a-b} \times gbx - ag^2y - abby} - \frac{aey^2 - aex^2 - \overline{fy - ax} \times e}{a^2 + ef \times x + \overline{e-f} \times ay}$$

$: a - c$

$: a - c - d - \frac{a^2y - a^2x^2 - \overline{fy - ax} \times ca}{ef + a^2 \times x + \overline{e - f} \times ay} - \frac{abdby}{\overline{a - b} \times gbx - abhy}$. Proinde ducendo media & extrema in ſe mutuo prodibit æquatio, cujus termini erunt x^4, xy^3, x^2y^2, xy^3, y^4, x^3, xy^2, x^2y, y^3 cum coefficientibus ſuis; unde patet Lineam eſſe quarti Ordinis.

Corol. I. Evaneſcente y æquatio erit hujus formæ $Ax^4 + Bx^3 = o$, cujus radices quatuor ſunt o, o, o, $-\frac{B}{A}$. Unde curva ter tranſibit per punctum C, adeoque curvæ hac *Prop.* deſcriptæ habent punctum triplex in C.

Corol. II. Revolvatur circa Polum C angulus datus TCO = LNT, & crurum CT ac SR concurſus ducatur per rectam BT, & crurum RH ac CO concurſus deſcribet Lineam tertii Ordinis quæ occurrere poteſt datæ AQ in tribus punctis, & proinde ter N coincidet cum C; eritque C punctum curvæ triplex, & in eo erit *Nodus* trium curvæ arcuum ſe mutuo ſecantium. Si vero AN dictam curvam contingat curva erit cuſpidata in C, & ſimul præter duos arcus qui cuſpidem formant tertius per idem punctum tranſibit; quod ſi AN in uno ſolo puncto dictæ curvæ tertii Ordinis occurrat, curva ſemel tranſibit ſimpliciter per C, & præterea ibi habebit punctum conjugatum. Univerſaliter rectæ quæcunque ex C eductæ in unico puncto præter C curvæ occurrunt.

Corol. III. Quod ſi anguli dati TCO, TRH ſint ejus magnitudinis ut dum moventur ſecundum *Corol.* 2. ſimul coincidant CO & RQ cum CR, curva deſcripta hac *Prop.* erit Linea tertii Ordinis, & C punctum ejus duplex.

Schol. Ex his *Prop.* intelligi poteſt quam vaſta curvarum multitudo ad quartum hunc Linearum Ordinem pertineat; Lineæ *Prop.* 14. deſcriptæ duo habent puncta duplicia, eæ *Prop.* 19. methodo deſcriptæ unicum habent punctum triplex, aliæ quoque poſtea delineandæ tria habent puncta duplicia, aliæ unicum punctum duplex, atque aliæ denique nulla niſi puncta ſimplicia; *Puncta* vero hæc poſſunt eſſe Nodata, Cuſpidata, Conjugata, vel nonnunquam partim Cuſpidata partim Nodata. Harum etiam curvarum aliæ ſunt pure Hyperbolicæ & crura habent octo, quatuor vel duo tantum pure Hyperbolica; aliæ ſunt Hyperbolo-Parabolicæ & duo habent crura Parabolica cum quatuor Hyperbolicis; aliæ ſunt pure Parabolicæ & habent quatuor vel duo crura pure Parabolica; & denique aliæ in ſe redeunt & in infinitum nequaquam abeunt. Sed nova quoque ex diametrorum numero & crurum poſitione oriuntur curvarum hujus Ordinis diſcrimina. Proinde enumeratio Linearum quarti Ordinis præclaræ *Neutonianæ* Linearum Ordinis tertii ſimilis abſque immenſo calculo nequit obtineri. Sed his ſæculis quibus feliciſſimo

Virorum doctorum ſtudio Artes ac Diſciplinæ omnes elegantiores, & præſertim Geometria, ad perfectionem ſummam properare videntur, ſperare licet Lineas quarti Ordinis non tam diu latere poſſe extra definitos Geometriæ limites quam prius latuerunt eæ ordinis proxime inferioris non ita pridem ab ipſo Geometrarum Principe in lucem proditæ.

Dixi Lineas quarti Ordinis tria poſſe habere puncta duplicia; ea vero nequeunt eſſe in eadem recta Linea, alias ea recta Lineam quarti Ordinis ſecaret in ſex punctis, quod abſurdum eſt. Linea quarti Ordinis non poteſt habere puncta quinque duplicia; quippe ſi haberet quinque duplicia Linea ſecundi Ordinis ſeu ſectio Conica per ea tranſiret, ut ex *Prop.* 4. manifeſtum eſt, & proinde concurſus ſectionis Conicæ cum Linea quarti Ordinis eſſent decem; ſed non poſſunt eſſe plures octo ex natura Linearum. Et Linea quidem quarti Ordinis nequit puncta duplicia habere plura tribus; quippe ſi quatuor puncta Lineæ quarti Ordinis duplicia eſſent ſectio Conica per ea quatuor ducta Lineam quarti Ordinis ſecaret in octo punctis, cum punctum quodvis duplex pro duobus ſimplicibus haberi poſſit. Sed ſectio Conica per quinque quævis puncta deſcribi poteſt; ſumatur igitur punctum quodvis ſimplex dictæ Lineæ quarti Ordinis, & ſectio Conica duci poſſet per quatuor ejus puncta duplicia & id quintum ſimplex, adeoque ſectio Conica in ea hyp. ſecare poſſet Lineam quarti Ordinis in novem punctis: ſed id abſurdum eſt; proinde Lineæ quarti Ordinis puncta quatuor nequeunt eſſe duplicia.

Oſtendimus *Prop.* præcedentibus qua ratione Lineæ quarti Orninis duci poſſunt, quæ duo puncta habent duplicia vel unicum punctum triplex; methodos ſimiles quibus eæ deſcribuntur quarum puncta tria ſunt duplicia, unicum duplex vel omnia ſunt ſimplicia poſtea explicabimus; eaſque tanquam *Corolaria Prop.* univerſalium Sectionis ſequentis brevitatis gratia demonſtrabimus.

Corol. Generale.

Ex Propoſitionibus 14, 17 *&* 19. *univerſaliter colligi poteſt, ſi in plano dentur tres rectæ* (Fig. 69.) AE, BF, DG *& dentur anguli* CNL, SRH, *& puncta duo* C *&* S *per quæ crura* CN *&* SR *ſemper tranſeunt, concurrant vero crura* CN *&* SR *in* T, *& crura* NL *ac* RH *in* Q, *& quadrilateri* NQRT *tria quævis puncta per datas rectas* AE, BF, DG *ducantur, punctum quartum deſcripturum Lineam quarti Ordinis.*

Sectio IV.

Ubi generalia demonſtrantur Theoremata de deſcriptione Linearum Ordinis cujuſcunque, Rectarum & Angulorum datorum ope.

POſtquam ſpecialiori Linearum curvarum trium primorum generum conſiderationi præcedentibus ſectionibus immorati ſumus, non opus eſt linearum altiorum ordinum examen adeo particulare inſtituere; eæ generaliter ſunt tractandæ & theoria noſtra Propoſitionibus aliquot infinite univerſalibus hac Sectione perficienda.

Lemma III.

Dentur poſitione rectæ (Fig. 70.) AN, DR, GL, &c. *anguli* SNR, NRL, RLT, &c. *& punctum* S; *ducantur* NU, LZ *normales in* CS *&* Tω *parallela rectæ poſitione datæ* CS, *occurrens ipſi* LZ *in* ω, *dico ſi quantitates quævis* x *&* y *ad unam tantum dimenſionem aſcendant in valore perpendicularis* NU, *rationem rectarum* Tω *&* Lω *exprimi poſſe quantitatibus in quibus eædem* x *&* y *erunt unius dimenſionis.*

Ex punctis N, R, L, &c. demittantur rectæ perpendiculares in CS ut NU, RH, LZ & ducatur recta Ro parallela ipſi CS occurrens normali NU in e, & LZ in o; ſit CS $= a$, SG $= g$, SD $= p$, SA $= b$, ſinus anguli SNR ad coſinum ut a ad f, ſinus anguli NRL ad coſinum ut a ad k, ſinus anguli RLT ad coſinum ut a ad m, & AU : NU :: $a : e$.

His poſitis cum ex hyp. x & y ad unam aſcendant dimenſionem in valore ordinatæ NU, patet ſi a, b, c, d, f, ſint quantitates datæ ordinatam exprimi poſſe hac generali æquatione NU $= \frac{y + ax + b}{cx + dy \mp f}$. Sed eſt SU $= b + \frac{a\text{NU}}{e}$, unde SU $= \frac{\overline{bcf + a^2} \times x + \overline{fbd + a} \times y + bfe + ab}{ecx + edy + ef}$. Sit $\overline{bcf + a^2} \times ax + \overline{fbd + a} \times ay + bfea + a^2b - fe \times \overline{ax + y \mp b} = t$, & $ae \times \overline{ax + y + b} + \overline{bcf + a^2} \times fx + \overline{fbd + a} \times fy + bf^2e + abf = u$.

Et

Et quoniam $SU - \frac{f}{a}NU : NU + \frac{f}{a} SU :: Ne : Re$ per ea quæ ſæpe prius demonſtrata ſunt, & ſpeciatim in principio demonſtrationis *Prop.* 5. erit $Ne : Re :: i : u$, & ſimili argumento oſtendi poteſt eſſe $Re + \frac{k \times Ne}{a} : Ne + \frac{k \times Re}{a} :: Lo : Ro$, unde $au + ki : ai + ku :: Lo : Ro$. Eadem ratione $Ro + \frac{m \times Lo}{a} : Lo + \frac{m \times Ro}{a} :: L\omega : T\omega$: unde $L\omega : T\omega :: \overline{ai + ku} \times a + m \times \overline{au + ki} : a \times \overline{au + ki} + m \times \overline{ai + ku}$. Sed in valoribus ſymbolorum i & u quantitates x & y ad unam tantum aſcendunt dimenſionem, & a, k, m, ſunt quantitates datæ, & proinde ratio $L\omega$ ad $T\omega$ exprimitur quantitatibus in quibus x, & y ad unam ſolam aſcendunt dimenſionem. Et ſi T ſit ſemper in recta IT & angulus LTX detur, ſit & Xr parallela CS ac Tr parallela NU, ſimiliter demonſtrari poſſet rationem Tr ad Xr exprimi poſſe quantitatibus in quibus x & y aſcendunt ad unam tantum dimenſionem.

LEMMA IV.

Si numerus rectarum datarum (Fig. 70.) AN, DR, LG, &c. *dicatur* (n) *& in valore* NU *ordinatæ ad primam rectam quantitates quævis* x, y *ad unam aſcendant dimenſionem, eæ ipſæ quantitates in valore ordinatæ* LZ *ad ultimam rectam* GL *erunt dimenſionum* (n).

Iiſdem ſymbolis manentibus ac in priori *Lemmate*, ſit præterea $DH : RH :: a : l$, & $GZ : LZ :: a : n$, dicatur $LZ = z$, & $RH = r$, eritque $SZ = g + \frac{az}{n}$, $SH = p + \frac{ar}{l}$, & $Lo = z - r$, $Ro = SZ - SH = g - p + \frac{az}{n} - \frac{ar}{l}$; ſed per *Lemma* 3. ratio Lo ad Ro exprimi poteſt quantitatibus in quibus x & y ſunt unius tantum dimenſionis, atque ſi $au + ki = t$ & $ai + ku = s$, erit $Lo : Ro :: t : s$, unde $z - r : g - p + \frac{az}{n} - \frac{ar}{l} :: t : s$, adeoque $z = \frac{nlt \times \overline{g - p} - aurt + lnrs}{lns - alt}$; quantitates n, l, n, g, p ſunt invariabiles & in valoribus quantitatum t, s ipſæ x & y unius ſunt dimenſionis, unde quantitates x & y in valore ordinatæ LZ $(= z)$ erunt dimenſionis unitate altioris quam in valore ordinatæ RH $(= r)$; & eodem argumento conſtat quantitates x, y aſcendere in valore ordinatæ RH ad dimenſionem unitate altiorem quam eſt earum dimenſio in valore ordinatæ NU; proinde cum in valore ipſius NU ſint

ſint x, y unius dimenſionis ex hypotheſi, erunt duarum dimenſionum in valore ordinatæ RH, atque trium in valore ordinatæ LZ ad tertiam rectam GL; proinde ſi augeatur numerus rectarum aſcendet dimenſio quantitatum x, y in valore ordinatæ ad ultimam rectam & ſi numerus rectarum ſit (n) quantitates x, y erunt dimenſionum (n) in valore ordinatæ ad ultimam rectam.

Corol. Duo hæc *Lemmata* obtinent, licet NU ſit ordinata ad Lineam quamvis Curvam ſi NU$=y$, & SU$=x$.

Prop. XX.

Cæteris manentibus ut in Lemmate præcedenti, detur aliud punctum C, *ex quo in* LT *demittatur ſemper* CP *angulum* CPL *datum conſtituens; moveantur puncta* N, R, L, *&c. per rectas ſuas* AN, DR, GL, *& punctum* P *interea deſcribet Lineam Ordinis* $n+1$. *ſi* n *denotet numerum datarum rectarum.*

Ex puncto P in datam rectam CS demittatur perpendicularis PM ordinata curvæ deſcriptæ ad axem CS, ex eodem puncto ducatur ad datam CS recta PT conſtituens angulum PTC æqualem dato CPL, ſit CE perpendicularis ex C in PT occurrens ordinatæ PM in F, ſitque Pω parallela axi CS occurrens rectæ LZ in ω, dicatur PM$=y$, CM$=x$, CS$=a$, LZ$=z$, & PM : TM :: a : t.

His poſitis anguli PTB, TPω & CPL erunt æquales, & proinde CPE $=$ LPω, adeoque ſimilia erunt triangula CPE, LωP ob angulos rectos CEP, PωL; proinde erit CE : PE :: Lω : Pω, ſed CE : CT :: PE : PF, unde CT : PF :: Lω : Pω, adeoque cum ſit ex hypotheſi recta MT$=\frac{ty}{a}$, MF$=\frac{tx}{a}$, L$\omega=z-y$ & P$\omega=$SM$-$SZ$=a+x-g-\frac{az}{n}$ erit $x+\frac{ty}{a} : y-\frac{tx}{a} :: z-y :: a+x-g-\frac{az}{n}$, unde prodit

$$LZ=z=\frac{\overline{a+x-g}\times\overline{anx+nty}+any^2-tnxy}{a\times\overline{ax+ty}+nt\times\overline{ay-tx}}.$$

Porro per *Lemma* 3. erit Lω : Pω :: $a\times\overline{ai+ku}+m\times\overline{au+ki}$: $a\times\overline{au+ki}+m\times\overline{ai+ku}$; ſed per idem *Lemma* ſi NU$=r$ erit

$i=b$

$\frac{i}{ae}=b+\frac{ar}{e}-\frac{fr}{a}$ & $\frac{u}{ae}=r+\frac{ar}{e}+\frac{fb}{a}$, unde $ax+ty:ay-tx$ (: : $L\omega : P\omega$) : :

$$a\times\overline{ab+\frac{a^2r}{e}-fr+kr+\frac{kfr}{e}+\frac{kfb}{a}}+m\times\overline{ar+\frac{afr}{e}+fb+kb+\frac{kar}{e}\,\frac{kfr}{a}}$$

$$:a\times\overline{ar+\frac{afr}{e}+fb+kb+\frac{kar}{e}-\frac{kfr}{a}}+m\times\overline{ab+\frac{a^2r}{e}-fr+kr+\frac{kfr}{e}+\frac{kfb}{a}}$$

ex qua analogia deduci poteſt valor ordinatæ NU ($=r$) in quo x & y erunt unius tantum dimenſionis; proinde per *Lemma* 4. quantitates x, y in valore ordinatæ LZ erunt dimenſionum (n) adeoque valor ipſius LZ erit hujus formæ $LZ=\frac{Ax^n+Bx^{n-1}y+Cx^{n-2}y^2 \text{ \&c.}}{ex^n+fx^{n-1}y+gx^{n-2}y^2 \text{ \&c.}}$ Sed x & y in prius reperto valore ordinatæ LZ erant duarum dimenſionum, & proinde ſi duo valores æquentur, & denominator unius ducatur in numeratorem alterius prodibit æquatio dimenſionum $n+2$ hujus formæ, $an\times ex^n y^2-tnex^{n+1}y+anex^{n+2}$ &c. $=Aa^2x^{n+1}$ &c. Ubi termini notantur tantum altiſſimi cum ii ad ordinem curvæ determinandum ſufficiant. Cum igitur in æquatione hac x & y ad dimenſionem aſcendant $n+2$, patet Lineam deſcribi Ordinis $n+2$.

Corol. I. Si ex puncto dato C in rectam RL demittatur ſemper CI conſtituens angulum CIR æqualem dato RLP, punctum I erit in Linea ordinis $n+1$ (cu m negligatur in hac deſcriptione ultima recta GL) quæ occurret rectæ GL in punctis $n+1$; horum duo ſint Λ, λ, & in deſcriptione hujus *Prop.* cum L pervenerit ad Λ vel λ, recta LP tranſibit per C, cum CI (quæ nunc evadit $C\lambda$) ſit ex Hypotheſi ſemper parallela rectæ LP; proinde coincidet P cum C, & curva tranſibit per punctum C; atque ubi L pervenerit ad aliud quodvis punctum rectæ GL in quo occurrit dictæ curvæ ordinis $n+1$, curva tranſibit per punctum C; unde manifeſtum eſt arcus curvæ $n+1$ tranſire per idem punctum C, quod proinde erit punctum curvæ multiplex; atque idem numerus qui exprimit ordinem curvæ, ſi unitate minuatur exprimet genus puncti C, ſeu quot arcus in eo concurrant.

Corol. II. Curvæ Propoſitione deſcriptæ *Cuſpidatæ* ſunt quoties recta GL dictam curvam ordinis $n+1$ contingit, & tot poteſt habere *Cuſpides* quot ſunt unitates in numero $\frac{n+1}{2}$; quod ſi DR curvam hic contingat illic ſecet, curva erit ſimul *Nodata* & *Cuſpidata* in puncto C.

Corol. III. Si demittatur ex C in SN recta CP ad datum quemcunque angulum, cum in hoc caſu n evaneſcat, punctum P deſcribet Lineam Ordinis ſecundi; quæ quidem circulus eſt ut ex Elementis conſtat. Si $n=1$, & CP demittatur in NR ad datum angulum, linea deſcripta erit Ordinis

Ordinis tertii, & quidem Hyperbola erit defectiva ut ex *Lemmate* 2. manifeſtum eſt; atque hac *Prop.* curvæ ſimpliciſſimæ Ordinis cujuſcunque deſcribuntur, cum vero methodus hæc ad paucas tantum Lineas ſeſe extendat ei non immorabimur ſed ad generaliora progrediemur.

Prop. XXI.

Cæteris manentibus ut in Lemmate 3. (Fig. 72.) *detur Poſitione alia recta* BQ, *angulus* FCO, & *punctum aliud* C; *circa quod ut Polum moveatur angulus* FCO, & *concurſus crurum* SN, CF *ducatur per rectam* BQ, *ſi puncta* N, R, L, *rectas ſuas infinitas percurrant concurſus crurum* CO & LT *deſcribet Lineam Ordinis* $n+1$, *ſi* n *denotet numerum Rectarum omnium poſitione datarum.*

SYmbolis Lemmatis 3 & 4 manentibus ſit præterea CB $= b$, PM $= y$, CM $= x$. Similia ſunt triangula LωT & LnP, adeoque per *Lemma* 3. erit Ln : Pn :: $a\times\overline{ai+ku}+m\times\overline{au+ki} : a\times\overline{au+ki}+m\times\overline{ai+ku}$, ſed L$n$ = LZ $-\,y$ & Pn = SM $-$ SZ $= a+x-g-\frac{a\text{LZ}}{n}$, unde

$$a\times\overline{ai+ku}+m\times\overline{au+ki} : a\times\overline{au+ki}+m\times\overline{ai+ku} :: \text{LZ}-y : a+x-g-\frac{a\text{LZ}}{n},$$

$$\& \ \text{LZ} = \frac{\overline{na+nx-ng}\times\overline{a\times\overline{ai+ku}+m\times\overline{au+ki}}+ayn\times\overline{au+ki}+nmy\times\overline{ai+ku}}{an\times\overline{au+ki}+nm\times\overline{ai+ku}+aa\times\overline{ai+ku}+am\times\overline{au+ki}}$$

in qua æquatione patet x & y ad duas tantum aſcendere dimenſiones, cum ipſæ reperiantur hic ſimpliciter, & in valoribus ſymbolorum i & u (per *Lemma* 3.) unius ſint dimenſionis. Et ſimilis inveniri poteſt valor ordinatæ ultimæ in ſerie quavis data, in quo x & y erunt ſemper duarum dimenſionum.

Sed per *Lemma* 4. alius indagatur valor ordinatæ LZ in quo x & y ad tot aſcendunt dimenſiones quot ſunt rectæ AN, DR, GL. &c. Proinde erit per *Lemma* 4. (cum omiſſa BQ numerus rectarum AN, DR,

GL ſit $n-1$) $LZ = \frac{ax^{n-1} + bx^{n-2}y + cx^{n-3}y^2, \&c.}{Ax^{n-1} + Bx^{n-2}y + Cx^{n-3}y^2, \&c.}$ In utroque valore terminos ſcripſimus ſolos altiſſimos cum ii ad ordinem curvæ determinandum ſufficiant.

Æquentur duo valores & prodibit curvæ æquatio hujus formæ,

$$a\,Ex^{n+1} + b\,Ex^n y + c\,Ex^{n-1} y^2 = e\,Ax^{n+1} + e\,Bx^n y, \&c.$$

unde manifeſtum eſt Lineam deſcribi Ordinis $n+1$.

Corol. I. Si in RL cadant rectæ ex C ad angulos æquales dato RLP, concurſu earum cum RL deſcribetur per *Prop.* 20. curva ordinis n (non $n+2$ quoniam BQ & GL in deſcriptione hac omittuntur) & quoties hæc curva ſecabit rectam DR toties curva tranſibit per C; proinde curvæ arcus (n) concurrunt in C.

Corol. II. Curvæ hæ cuſpidatæ ſunt in C quoties GL dictam curvam contingit, & tot habere poſſunt *Cuſpides* quot ſunt unitates in numero $\frac{n}{2}$; quod ſi DR curvam hic contingat, illic ſecet, curva erit ſimul *Nodata* & *Cuſpidata* in C.

Corol. III. Producatur LP donec occurrat rectæ SN productæ in e, & rectæ CQ in Δ; cum dentur anguli SNR, NRL, RLP manifeſtum eſt dari angulum $Se\Delta$. Unde ſi ſuper CS deſcribatur circulus, cui inſcribatur angulus æqualis ſupplemento datorum FCO & $Se\Delta$ ad quatuor rectos, occurrens datæ rectæ BQ in punctis K & q, & anguli QCP, qCπ æquales ſint dato FCO, erunt rectæ CP, Cπ parallelæ duabus curvæ Aſſymptotis.

Corol. IV. Dum recta LP tranſit per C, ſi ſimul coincidant CO cum CL, curva erit Linea ordinis tantum n, & ſi in alio quoque caſu quo tranſit LP per C coincidat CO cum CL, curva erit ordinis $n-1$; unde *Prop.* hæc non caſus tantum exhibet quibus lineæ ordinis cujuſcunque delineari poſſunt, ſed ſinguli ejus caſus methodos ſuppeditant particulares curvas aliquas ordinis omnis inferioris eo qui in *Prop.* exprimitur deſcribendi.

Corol. V. Concurrant rectæ AN, DR productæ in E, & ſi angulus SNR cum SER conficiat duos rectos, Linea deſcribetur ordinis n. Nam per *Lemmatis* 2. *Corol.* dabitur angulus SRL; & proinde eadem deſcribetur curva ac ſi omiſſa recta AN, loco anguli NRL ſubſtituatur datus SRL, unde manifeſtum eſt ordinem curvæ evadere unitate inferiorem; ſimiliter ſi concurrant GL & DR in H, & SRL ſupplementum ſit dati SHL ad duos rectos dabitur angulus SLT, & curva erit ordinis $n-1$.

Corol. VI. Eadem deſcribi poteſt curva ſi cæteris manentibus ut in *Prop.* loco anguli FCO revolvatur radius rectus circa Polum C, ſubſtitutis

ſtitutis loco BQ & AN aliis quidem rectis quarum Poſitio ex *Prop.* 7. & 3. intelligi poteſt. Quippe omnes curvas hac Propoſitione deſcriptas faciliori illo modo poſſe conſtrui eadem ratione demonſtratur qua ſpecialem ejus caſum *Prop.* 7. demonſtravimus.

Corol. VII. Cum Linea deſcripta ſit ordinis $n+1$, hinc ſi $n=3$ erit Linea ordinis quarti, & curva habebit punctum triplex; unde alia prodit methodus Lineas quarti Ordinis deſcribendi quæ punctum habent triplex præter eam quam demonſtravimus *Prop.* 19.

PROP. XXII.

Iiſdem poſitis ac in Prop. *præcedente, dico ſi omnia puncta* (Fig. 73.) Q, N, R, L, P, *&c. quorum numerus eſt* $n+1$, *præter unum quodvis, ducantur per rectas infinitas* BQ, AN, DR, GL, EP, *&c. illud unum deſcripturum Lineam Ordinis* $n+1$.

SUpponamus puncta Q, N, R, P rectas ſuas percurrere & inveſtigandum ſit quænam deſcribatur Linea motu puncti angularis cujuſvis L. Ut id conſtet ducatur recta quævis GL, & ſi huic rectæ curva occurrat in punctis $n+1$, neque aliam quamvis rectam in plano ductam in pluribus punctis ſecare poſſit, Linea neceſſario deſcribetur ordinis $n+1$, & conſtabit Propoſitio.

Moveantur imprimis puncta omnia Q, N, R, ut in *Prop.* 21. percurrat punctum L rectam GL, & linea deſcribetur motu puncti P ordinis $n+1$, occurrens rectæ EP in punctis totidem; quorum duo aliqua ſint Π, π, quæ loca ſint puncti P cum L pervenit ad Λ & λ; & ſimiliter erunt puncta $n+1$ in recta GL reſpondentia punctis $n+1$ quibus recta EP occurrit curvæ deſcriptæ ſecundum *Prop.* præcedentem. Proinde ſi nunc P concurſus crurum CO & LT ducatur per rectam infinitam EP, perveniat vero P ad Π & π punctum L ſimul accedet ad Λ & λ; & dum P verſatur in dictis punctis $n+1$, quibus recta EP occurrit curvæ deſcriptæ ſecundum *Prop.* præcedentem, punctum L eodem tempore erit in punctis $n+1$ correſpondentibus rectæ GL; & quoniam curvæ dictæ recta EP non poteſt occurrere in pluribus punctis quam $n+1$, patet punctum deſcribens L non in pluribus locis exiſtere in

recta GL dum curvam deſcribit. Curva igitur motu puncti L deſcripta non poteſt occurrere rectæ cuivis in plano ductæ in punctis pluribus quam $n+1$; aſt in tot rectam GL ſecare poteſt, adeoque Linea deſcripta erit ordinis $n+1$. Sed cum nihil ſit aſſumptum de puncto L quod de reliquis pari jure dici non poſſit, patet eorum quodvis deſcripturum Lineam ordinis $n+1$ ſi reliqua ducantur per rectas ſuas infinitas. Unde Propoſitio univerſaliter conſtat.

Corol. I. Hinc tot modi deducuntur Lineas ordinis cujuſcunque deſcribendi quot unitates ſunt in numero qui exprimit ordinem curvæ.

Corol. II. Si curva deſcribatur motu puncti Q ea habebit punctum ſimplex in C, ſed totuplex erit punctum S quotuplex erat C in *Prop.* præcedente; unde tertia prodit methodus Lineas quarti Ordinis deſcribendi quæ punctum habent triplex. Plurima *Prop.* præcedentis *Corolaria* hic quoque obtinent univerſaliter.

Prop. XXIII.

Cæteris manentibus ut in Prop. 21. *ducatur concurſus crurum* (Fig. 74.) CF *&* RL *per rectam infinitam* BQ, *& ſi numerus* (r) *exprimat ordinem rectæ* RL *in ſerie rectarum* SN, NR, RL, LT, *&c. concurſus crurm* CO *&* LT *deſcribet Lineam Ordinis* $n+r$.

Ut Lineæ deſcriptæ ordo innoteſcat ducatur recta quævis KX, & inveſtigandum ſit quoties curva occurrere poteſt huic rectæ; ſupponamus concurſum crurum CO & LT duci per rectam KX & crurum CF, SN concurſus deſcribet Lineam ordinis $n+1$ per *Prop.* præcedentem, quæ tranſibit ſemel per C, & toties per S quoties unitas reperitur in numero *(n)*. Dein ſupponamus concurſum crurum CF, RL duci per rectam infinitam BQ, & per eandem *Prop.* concurſus crurum CF, SN deſcribet Lineam ordinis $r+1$, cujus arcus *(r)* tranſibunt per S, unicus vero arcus tranſibit per C.

Sed in deſcriptione hujus Propoſitionis concurſus crurum CF, RL ducitur per rectam BQ, adeoque punctum P tunc erit in recta KX cum concurſus crurum CF & SN pervenerit ad occurſum aliquem duarum curvarum quas deſcripſimus, quarum altera eſt ordinis $n+1$ altera ordinis $r+1$. Sed ex natura Linearum ii concurſus poſſunt eſſe $nr+n+r+1$,

quorum

quorum nr fiunt in puncto S, & unus in ipſo C; cum prioris curvæ puncta (n) ſint in S, & poſterioris puncta r ſint in eodem, utraque vero ſemel tranſeat per C. Atqui concurſus crurum CF, SN in S vel C non ſunt conſiderandi & reliqui ſunt $n+r$; ex puncto S ad eos concurſus ducantur rectæ SN, & eæ ſecabunt datam AN in punctis $n+r$; ad ea puncta ſuper dictas rectas ex S eductas conſtitue angulos æquales dato SNR, & rectæ NR ſecabunt datam DR in totidem punctis; ducantur dein rectæ totidem RL & LT, & hæ ſecabunt datam KX in punctis $n+r$, quæ puncta erunt curvæ hac *Prop.* deſcriptæ. Proinde curva occurrit rectæ KX in punctis $n+r$, atque adeo erit Linea ordinis $n+r$.

Hæc *Prop.* alia quoque ratione ex *Lemmate* 4. demonſtrari poteſt. Ducatur enim concurſus crurum CF & RL per rectam BQ, & concurſus crurum CO & LT per rectam datam PU, puncta angularia L, K, &c. per rectas infinitas & punctum N deſcribet Lineam Ordinis $n+r$. Quippe cum $NU=y$, & $SU=x$, manifeſtum eſt *Lemmate* 4. x, y aſcendere ad dimenſionem r in valore ordinatæ ad rectam BQ; ſed ad eandem aſcendunt poteſtatem in valore ordinatarum ad rectas BQ & PU ob datum angulum QCP; proinde x & y in valore ordinatæ PM erunt dimenſionum r, ſed per idem *Lemma* x, y in valore ordinatæ ad ultimam rectam PU in ſerie AN, DR, GL, PU erunt dimenſionum n. Duo igitur prodeunt valores ordinatæ PM in quibus x, y aſcendunt ad dimenſiones n & r; hi vero valores ſunt fracti, & proinde ſi æquentur prodibit æquatio in qua x & y erunt dimenſionis $n+r$; adeoque linea motu puncti N deſcripta erit ordinis $n+r$, quæ occurrere poteſt rectæ AN in punctis $n+r$. Sed in deſcriptione hujus *Prop.* quoties N pervenerit ad occurſus rectæ AN cum Linea dicta ordinis $n+r$, punctum P concurſus crurum CO & LT tanget rectam PU; unde manifeſtum eſt Lineam motu puncti P deſcriptam rectæ in plano ductæ occurrere poſſe in punctis $n+r$, ſed non in pluribus, adeoque Lineam deſcribi ordinis $n+r$.

Corol. I. Curvæ arcus (n) concurrunt in puncto C. Quippe ſi ex C in RL cadant rectæ CZ ſemper parallelæ rectæ LT, adeoque conſtituentes angulum CZR æqualem dato RLT, per *Prop.* 20. punctum Z erit ſemper in Linea Ordinis (n) quæ rectam GL ſecare poterit in punctis (n), ſed in deſcriptione hujus *Prop.* ubi punctum L accedit ad concurſum rectæ GL cum dicta linea coincidunt L & Z, adeoque etiam LT & ZC, & proinde P coincidet cum C. Unde curva hac *Prop.* deſcripta habebit puncta (n) in Polo C.

Corol. II. Recta quævis ex C educta curvæ occurrere poteſt in punctis (r) præter ipſum C; quippe detur recta quævis CO, & quæratur quoties curva huic rectæ occurrere poſſit; cum detur CO dabitur etiam CF ob datum angulum FCO, & proinde datur punctum Q. Ex dato puncto Q ad rectam NR ſemper ducantur rectæ parallelæ ipſi RL, atque curva

hac

hac ratione deſcribetur ordinis (r) per *Prop.* 20. quæ occurret rectæ DR in punctis (r), & quoties in deſcriptione *Prop.* hujus punctum R pervenerit ad occurſus curvæ dictæ cum DR curva deſcripta motu puncti P ſecabit rectam CO; nam ubi R pervenerit ad puncta ubi linea ea ordinis (r) ſecat rectam DR, punctum L accedet ad totidem puncta iis reſpondentia in recta GL, adeoque dum unica data manet poſitio rectæ CO, recta LT poteſt varias obtinere poſitiones quarum numerus eſt r; adeoque rectæ CO occurre poteſt in punctis (r).

Corol. II. Demonſtravimus *Corol.* 1 & 2 abſque *Prop.* hujus ope; ex *Corol.* I. manifeſtum eſt Lineam *Prop.* hac deſcriptam habere puncta (n) in dato puncto C; oſtendimus *Corol.* 2. eandem Lineam ſecare rectam ex C eductam in punctis(r) præter ipſum punctum C; unde patet rectam ex C eductam curvæ occurrere in punctis $n+r$; atque hinc *Prop.* confirmatur; cum hoc quoque ratiocinio prodeat ordo curvæ $n+r$.

Corol. III. Quemadmodum *Prop.* 22. ex 21 demonſtravimus ex hac oſtendi poſſet univerſaliter, ſi omnia puncta N, R, L, Q, P ducantur per rectas ſuas indefinitas præter unum quodvis, id unum deſcripturum Lineam ordinis $n+r$.

Corol. IV. Cæteris manentibus ut in *Prop.* ſumantur in eodem plano (*Fig.* 75.) rectæ ΓΔ, ΘI, ΛΦ, &c. quarum ſit numerus $(r-1)$ occurrat ΓΔ rectæ SN in Γ ſintque anguli SΓΘ, ΓΘΛ dati, & ſi concurſus crurum CF & ΘΛ ducatur per rectam BQ dum N, R, L, Γ, Θ, &c. rectas ſuas infinitas percurrunt, dico concurſum cruris CO & LT, punctum ſcil. P, deſcripturum Lineam ordinis $n+r$. Corolarium hoc eadem ratione demonſtratur ac *Prop.*

Corol. V. Ex *Corol.* 3 & 4 manentibus n, & r deducuntur methodi diverſæ Lineas ordinis $n+r$ deſcribendi quarum numerus eſt $n+r$; ſed manente ſumma $n+r$ ipſi numeri n & r variari poſſunt diverſis modis $\frac{n+r}{2}$, ita ut diverſas ſemper methodos exhibeant lineas eas deſcribendi: Proinde ex hac *Prop.* deduci poſſunt modi $\frac{N^2}{2}$ deſcribendi curvas ordinis N ſi N ſit numerus par, vel $\frac{N \times \overline{N-1}}{2}$ ſi ſit impar. Harum vero aliquæ ſæpe coincidunt, & ſpeciatim ſi N $(=n+r)$ ſit numerus par methodi quæ prorſus diverſæ haberi poſſunt ex *Corol.* 3. deductæ, ubi $n=r$ ſunt tantum $\frac{n+r}{2}=\frac{N}{2}$, quas ſuppoſuimus eſſe $n+r$; proinde ex dicto numero $\frac{N^2}{2}$ ſubducatur $\frac{N}{2}$, & methodi diverſæ quæ ex *Corol.* 3 & 4

deduci

deduci poſſunt Lineas ordinis N deſcribendi erunt $\frac{N^2 - N}{2}$, ſive N numerus ſit par ſive impar.

Corol. VI. Si $N = 4$ erit $\frac{N}{2} \times N - 1 = 6$; proinde ex hac *Prop.* deduci poſſunt ſex methodi diverſæ quibus Lineæ quarti Ordinis deſcribuntur; harum quatuor ex præcedente *Prop.* deduci poſſunt ſumendo tres tantum rectas *(Fig. 68.)* AN, DR & BQ; nam quatuor punctorum Q, N, R, P unum quodvis deſcribet Lineam quarti Ordinis, ſi reliqua ducantur per rectas datas infinitas. Aliæ duæ methodi ex hac *Prop.* ejuſque *Corol.* 4. deducuntur; dentur tres rectæ *(Fig. 76.)* AN, DR & BQ, dentur anguli SNQ, SRT & FCO; ducatur concurſus crurum CF & NQ per rectam CQ, dum N & R percurrunt rectas ſuas infinitas AN & DR, atque concurſus crurum CO & RT deſcribet Lineam quarti Ordinis punctum duplex habentem in C. Super CS deſcribatur circulus habens angulum SRT ei inſcriptum, & ſi hic circulus bis occurrat rectæ DR curva habebit *Nodum* vel Crucem in C, ſi circulus rectam contingat curva erit *Cuſpidata* in C, & ſi recta ponatur tota extra circulum curva habebit *Punctum* conjugatum in eodem. Sed ſi coincidat CO cum RT dum RT tranſit per C, Linea deſcribetur tertii Ordinis, & punctum C erit ſimplex; atque hinc Lineæ tertii Ordinis aliquæ quæ puncto duplice deſtituuntur deſcribi poſſe videntur. Si puncta Q, N, P ducantur per rectas datas punctum angulare R deſcribet Lineam quarti Ordinis. Atque hæ ſex ſunt methodi quibus Lineæ hujus Ordinis deſcribuntur præter tres *Prop.* 14, 17, 19 demonſtratas. Lineæ motu punctorum N vel Q deſcriptæ, non ſuppeditant methodos diverſas ab iis quas explicuimus.

Corol. VII. Si $N = 5$ erit $\frac{1}{2} \times \overline{N^2 - N} = 10$, unde decem modi deſcribendi Lineas quinti Ordinis ex *Prop.* deduci poſſunt, & ſi $N = 6$ erit $\frac{1}{2} \times \overline{N^2 - N} = 15$.

PROP.

Prop. XXIV.

Sumantur ad libitum rectæ (Fig. 77.) *in eodem plano ubicunque poſitæ* BN, ER, FT *quarum ſit numerus (n). Sumantur etiam ad libitum aliæ rectæ* DM, GL, HK *quarum ſit numerus (m) ſint anguli* CNR, NRT, RTQ, *&c. atque anguli* SML, MLK, LKQ, *&c. invariati; dentur puncta* C *&* S, *at puncta angularia* N, R, T, M, L, K *ducantur per rectas infinitas* BN, ER, FT, DM, GL, HK, *& concurſus crurum* TQ *&* KQ *per rectam* AQ; *dico concurſum crurum* CN *&* SM *deſcripturum Lineam Ordinis* $n+m+2$.

HÆC ex iis quæ *Prop.* 21. demonſtravimus facile conſtat. Quippe ſi PM $= y$, CM $= x$, CB $= c$, CS $= a$, BU : NU :: $a : d$, erit NU $= \frac{dcy}{dx - ay}$, & CU $= \frac{dcx}{dx - ay}$. Unde cum x & y reperiantur ſimplices in valore ordinatæ NU ad rectam primam BN, erunt dimenſionis $n+1$ in valore ordinatæ ad ultimam rectam AQ, cujus proinde valor erit $\frac{Ax^{n+1} + Bx^n y, \&c.}{ax^{n+1} + bx^n y, + \&c.}$ ubi notantur tantum termini in quibus indices quantitatis x ſunt altiſſimi.

Similiter Mz ordinata rectæ DM deprehenditur æqualis $\frac{efy}{e \times \overline{a - x} - ay}$, ſi SD $= f$, CS $= a$, Dz : Mz :: $a : e$. Cumque x & y ſint unius dimenſionis in valore ordinatæ ad primam rectam, & rectarum numerus ſi includatur AQ ſit $m+1$, patet x & y aſcendere ad dimenſionem $m+1$ in valore ordinatæ ad ultimam rectam AQ, cujus proinde valor ſecundus erit $\frac{Ex^{m+1} + Fx^n y, \&c.}{ex^{m+1} + fx^m y, \&c.}$

Æquentur

Æquentur duo reperti valores ordinatæ rectæ AQ, eritque $Aex^{m+2+n} + Afx^{m+n+1}y$, &c. $= aEx^{m+n+2}$, &c. Unde cum index altiſſimus quantitatum variabilium ſit $n+m+2$ patet lineam deſcribi ordinis $n+m+2$. Vel ſi numerus omnium rectarum, ſit N curva erit Linea ordinis $N+1$; ita ut genus curvæ ſemper exprimatur numero datarum rectarum.

Corol. I. Curvæ arcus $n+1$ concurrunt in puncto C, & arcus $m+1$ concurrunt in puncto S, nec alibi quam in C & S curva poteſt occurrere rectæ CS; hæc demonſtrantur ex *Prop.* 20. ad modum *Corol.* 1. *Prop.* 21.

Corol. II. Quemadmodum ex *Prop.* 21. demonſtravimus 22. ex hac demonſtrari poſſet univerſaliter, ſi omnia puncta N, R, T, Q, K, R, L, M, P ducantur per rectas ſuas indefinitas præter unum quodvis, id unum deſcripturum Lineam ordinis $n+m+2$, & manente numero rectarum, quodcunque ſit punctum cujus motu curva determinatur, manere ordinem lineæ deſcriptæ.

Corol. III. Manente ſumma $m+n$ manet ordo curvæ, ſed manente ſumma $m+n$ ipſi m & n variari poſſunt modis $\frac{m+n}{2}$, ita ut in ſingulis hypotheſibus C & S ſint diverſi generis puncta. Sed ipſis m & n manentibus modi ex *Corol.* præcedente deduci poſſunt curvas ordinis ejuſdem deſcribendi quorum numerus eſt $n+m+2$. Unde ſi N exprimat ordinem curvæ, diverſæ methodi quæ ex hac *Prop.* deduci poſſunt quibus curvæ ejus ordinis deſcribuntur erunt $\frac{1}{2}N \times \overline{N-2}$ ſi N ſit numerus par, vel $\frac{1}{2}N \times \overline{N-3}$ ſi ſit impar. At ſi N ſit numervs par, cum m & n fiunt æquales methodi prodeunt diverſæ non $n+m+2$ ſed $\frac{1}{2}\overline{n+m+2} = \frac{1}{2}\overline{N+1}$. Proinde ſubducendi ſunt $\frac{1}{2}N-1$ ex dicto numero $\frac{1}{2}\times\overline{N^2-2N}$, & methodi hujus *Prop.* diverſæ erunt $\frac{1}{2}N\times\overline{N-3}+1$; adjunge methodos *Corollarii* 5. *Prop.* præcedentis, & ſumma N^2-2N+1 oſtendet omnes methodos quas adhuc exhibuimus deſcribendi Lineas ordinis N ſi N ſit numerus par; aſt N^2-2N oſtendit methodos omnes quæ ex *Prop.* præcedentibus deduci poſſunt deſcribendi Lineas ordinis imparis. Unde ſi ordines Linearum exprimantur numeris 3, 4, 5, 6, &c. methodi omnes erunt 3, 9, 15, 25, &c.

Corol. IV. Si in hac *Prop.* cæteris manentibus curva non determinetur concurſu rectarum CN & SM, ſed aliarum rectarum quæ cum CN & SM datos conſtituunt angulos ad puncta C & S, curva tamen non alia deſcribetur quam quæ methodo hujus *Prop.* deſcribi poteſt per *Lemma* 1.

Corol. V. Si ſimul coincidant SP & CP cum CS curva deſcribetur ordinis inferioris quam ſecundum Propoſitionem eſſet. Atque hinc modi indagari poſſunt lineas deſcribendi puncto quovis duplice aut multiplice deſtitutas cujus exemplum exhibuimus *Prop.* 15.

Prop. XXV.

Cæteris manentibus ut in Prop. 24. (Fig. 78.) *concurrant rectæ* RT *&* SM *in* O, *& ſi puncta* N, R, T, Q, K, L, M, O *omnia ducantur per rectas poſitione datas præter unum quodvis, dico id unum deſcripturum Lineam Ordinis* $ms+s+n+1$ *ſi s denotet ordinem Lineæ* RT *in ſerie rectarum* CN, NR, RT, TQ.

MOveantur imprimis omnia puncta R, T, Q, K, L, M, O per rectas datas ER, FT, AQ, HK, GL, DM, OY & oſtendam Lineam motu puncti angularis N deſcriptam eſſe ordinis $ms+s+n+1$; unde facile erit *Prop.* univerſalem reddere.

Cum ordinata curvæ ad axem CS ſit NU $=y$, & abſciſſa ſit CU $=x$, erunt per *Corol. Lemmatis* 4. x & y dimenſionis s in valore ordinatæ ad rectam OY; ſed ſi ordinata rectæ OY dicatur u, ordinata rectæ DM $=z$, SY $=f$, SD $=b$, ſinus anguli SYO ad coſinum ut a ad d, & ſinus anguli SDM ad coſinum ut a ad e erit $z=\frac{dbeu}{edf-eau+dau}$, unde quantitates x, y erunt dimenſionis s in valore ordinatæ ad rectam DM. Sed per *Lemma* 4. erit (u) ordinata rectæ DM dimenſionis $m+1$ in valore QI ordinatæ ad rectam AQ, & proinde quantitates x, y erunt dimenſionum $ms+s$ in valore eodem.

Sed alius ex eodem *Lemmate* prodit valor QI ordinatæ ad rectam AQ ex conſideratione rectarum ER, FT, AQ & angulorum datorum CNR, NRT, RTQ in quo x & y erunt dimenſionis $n+1$. Æquentur duo hi valores ordinatæ QI ad rectam AQ, cumque ſint fracti ducatur Numerator unius in Denominatorem alterius, & prodibit æquatio in qua ſolæ quantitates variabiles x & y erunt dimenſionum $ms+s+n+1$; ſed y eſt ordinata curvæ motu puncti N deſcriptæ & x ejuſdem abſciſſa, & proinde ea linea curva erit ordinis $ms+s+n+1$.

Siquidem vero punctum N deſcribat Lineam ordinis $ms+s+n+1$, eadem ratione qua *Prop.* 22 ex 21 demonſtravimus, oſtendi poteſt Lineam motu puncti cujuſvis omnium N, R, T, Q, O, K, L, M deſcriptam, dum reliqua rectas poſitione datas percurrunt, fore ejuſdem ordinis.

Corol. I.

Corol. I. Curvæ arcus $ms+s$ in eodem puncto S concurrunt, & recta quævis ex S educta occurret curvæ in punctis tantum $n+1$, hæc ope *Prop.* 20. facile demonſtrantur eadem fere ratione qua *Corol.* 1 & 2 *Prop.* 23. ex *Prop.* 20. deduximus.

Corol. II. Si $s=2$, $m=1$ & $n=0$ Linea erit Ordinis quinti; atque hæc ſimpliciſſima eſt deſcriptio Linearum hujus Ordinis, cum nec plures anguli nec rectæ ad eas deſcribendas neceſſariæ ſint quam ad Lineas ordinis quarti delineandas adhibuimus *Prop.* 14, 17 & 19. Curvæ hæ habent punctum quadruplex in S, & rectis ex S eductis in uno ſolo puncto occurrunt; eæ vero per C nequeunt tranſire.

PROP. XXVI.

Cæteris manentibus ut in Prop. 24. (Fig. 77.) *ſi (s) exprimat ordinem rectæ* RT *in ſerie rectarum* CN, NR, RT, TQ, *& (r) ordinem rectæ* LK *in ſerie* SM, ML, LK, KQ, *Linea deſcripta motu puncti* O *concurſus rectarum* RT, LK, *dum puncta* N, R, T, Q, K, L, M *percurrunt rectas infinitas datas* BN, ER, FT, AQ, HK, GL, DZ, *erit ordinis* $ms+nr+s+r$.

QUoniam punctum Q rectam AQ percurrit manifeſtum eſt ex *Prop.* 24. concurſum crurum CN, SM deſcripturum lineam ordinis $n+m+2$. Proinde ad ordinem lineæ motu puncti O deſcriptæ inveſtigandum, negligantur rectæ FT, AQ, HK, & ſupponamus concurſum crurum CN, SM duci per Lineam ordinis $n+m+2$ *Prop.* 24. deſcriptam, quæ puncta habet $n+1$ in C, & $m+1$ in S per *Corol.* I. ejuſdem Propoſitionis, & quæratur ordo lineæ deſcriptæ motu puncti O concurſus crurum RT, LK.

Ducatur recta quævis OY & inveſtigandum ſit quoties linea deſcripta motu puncti O occurrat rectæ OY; ut id conſtet ſupponamus concurſum crurum RT, LK duci per eam rectam OY, & concurſus crurum CN, SM deſcribet Lineam ordinis $r+s$ per *Prop.* 24, quæ occurret Lineæ ordinis $n+m+2$ in punctis $nr+ns+mr+ms+2r+2s$ ex natura Linearum. Lineæ vero ordinis $n+m+2$ puncta $n+1$ ſunt

in C, & puncta $m+1$ in S; Lineæ ordinis $r+s$ puncta s ſunt in C, & r in S; proinde harum Linearum occurſus $ns+s$ ſunt in C, & $mr+r$ in S; aliæ vero earum interſectiones poſſunt eſſe $ms+nr+r+s$. Sed cum concurſus crurum CN, SM ad eas interſectiones accedit erit concurſus crurum RT, LK in recta OY, & in nullo alio caſu in ea recta crura RT, LK poſſunt concurrere; ſed cum concurſus crurum CN, SM fit in C vel in S, rectæ RT, LK non poſſunt concurrere in recta OY; niſi forte Lineæ ordinis $n+m+2$ & $r+s$ ſe mutuo contingant in iis punctis C, S, quo in caſu duæ interſectiones abeunt in unum punctum contactus, & altera earum interſectio erit neceſſario aliqua earum quas non ſuppoſuimus eſſe in punctis C, S. Proinde interſectiones ſunt $ms+nr+r+s$ Linearum dictarum, ad quas cum accedit concurſus crurum CN, SM, rectæ RT, LK concurrunt in aliquo puncto rectæ OY; & plures ejuſmodi interſectiones non ſunt. Proinde Linea deſcripta motu puncti O ſecabit rectam OY in punctis $ms+nr+s+r$, & non in pluribus. Cumque idem de alia quavis recta in plano rectarum datarum ducta demonſtrari poſſit, liquet lineam motu puncti O deſcriptam eſſe ordinis $ms+nr+s+r$.

Corol. I. Hinc ſi $r=1$ erit Linea ordinis $ms+s+n+1$, & curva determinabitur concurſu rectæ RT cum SM, qui caſus eſt ſpecialis *Prop.* hujus univerſaliſſimæ quem *Prop.* præcedente alia ratione demonſtravimus; & is eſt quem propoſuimus in Actis Philoſophicis, N° 359.

Corol. II. In caſu quem figura deſignat ubi $n=m=s=r=3$ Linea deſcribitur ordinis 24.

Corol. III. Eadem ratione qua *Prop.* 22 ex 21 demonſtravimus oſtendi poteſt curvam ejus ordinis qui in *Prop.* hac explicatur deſcribi motu cujuſvis punctorum N, R, T, Q, K, L, M, O, ſi reliqua per rectas ſuas indefinitas ducantur, & anguli iidem ut prius maneant invariati.

Corol. IV. Curvæ ordinis infimi quæ hac *Prop.* deſcribuntur ſunt eæ Ordinis quarti; quæ deſcribuntur cum $m=n=s=r=1$. Si s vel r ſint numero binario æquales & $m=n=1$ curva erit Ordinis ſexti.

Corol. V. Si ſuper CS deſcribatur circulus cui inſcribi poſſit angulus datus æqualis angulo SML + MLR — CNR — NRT, hic circulus occurret rectis CN & SM in eodem puncto dum crura RT & LK evadunt parallela & curva abit in infinitum. Sed rectæ CN & SM per *Prop.* præcedentem concurrunt ſemper in curva quæ Linea eſt Ordinis $n+m+2$ quæque dicto circulo occurrit in punctis $2n+2m+4$, horum vero concurſum $n+1$ ſunt in C & $m+1$ in S, & reliqui ſunt $n+m+2$; ſed ubi P concurſus crurum CN & SM pervenit ad concurſum curvæ dictæ & circuli, crura RT & LK quorum concurſu curva hujus Propoſitionis determinatur evadunt parallela, & curva excurrit in infinitum. Proinde curva habebit Aſſymptotos $n+m+2$ parallelas rectis RT & LK ubi CN & SM concurrunt in circulo deſcripto ſuper CS habente

bente angulum prius determinatum ei inſcriptum. Curva vero plura habebit crura infinita quam quæ ſunt ad eas Aſſymptotos, nam ubi recta RT vel LK abit in infinitum curva quoque abit in infinitum, & alia formantur crura curvæ infinita.

Prop. XXVII.

Cæteris manentibus ut in Prop. 24. (Fig. 79.) *ſumantur ad libitum rectæ in eodem plano bn, er, &c. quarum numerus dicatur (s—1) & rectæ aliæ dm, gl, &c. quarum numerus ſit (r—1) dentur anguli* Snr, *nrt, &c. & anguli* Sml, *mlk, &c. ducatur concurſus rectarum* TQ *&* KQ *ut in* Prop. 24. *per rectam datam* BQ, *& concurſus crurum rt & lk deſcribet Lineam Ordinis* $ms+nr+s+r$.

SIT O concurſus crurum *rt*, *lk*, & ſi O ducatur per rectam quamvis OY in plano *figuræ* ductam, rectarum SN & CM concurſus deſcribet Lineam ordinis $r+s$ per *Corol.* 4. *Prop.* 24. Unde demonſtratio hic eadem ratione procedit ut in *Prop.* 26. quam non opus eſt repetere.

Scholium.

Propoſitionibus his Theoriam exhibuimus deſcriptionis Curvarum motu angulorum per lineas rectas adeo ſimplicem & univerſalem quam rei natura admittere videtur. Recta eſt omnium Linearum prima & in noſtris conſtructionibus poſtulatur; earum unius ope Linearum Ordinis ſecundi ſyſtema proxime ſimplex deſcripſimus; atque his duobus primis Linearum ordinibus concludebatur fere integra veterum Geometria; & raro ulterius provehebantur niſi ad unam aut alteram aſſumendam altioris ordinis ad ſolutionem Problematum difficiliorum obtinendam.

Recentiores novos quidem aperuerunt curvarum campos in infinitum diffuſos, & ardua ſatis Theoremata de earum menſuris variiſque proprietatibus demonſtrarunt, utpote eximiæ methodi Fluxionis ope præditi. Aſt plerique dum ſuas ſpeciales condunt Theorias, varium perpetuo adhibent curvas in ordines redigendi modum, primarii hujus propemodum immemores.

immemores. Sic inter viros illuſtres de re Geometrica qui bene meruerunt plures curvarum Hyperbolicos & Parabolicos novos conſtruxerunt ordines, quorum exemplo ducti ſimiles Spiralium claſſes compoſuerunt D. *Fermat* & *Varignon*; Cl. Geometra D. *de la Hire*, ut veteres conchoidem ex recta, ita ſpecies complexiores aliquas ex ſimplicioribus conſtruere docuit; D. *Nicol* curvas ſuper baſes ſibi æquales rotavit ad complexiores Geometricas delineandas & novo artificio Geometriam adauxit: Illi quidem aliique, quorum induſtriam laudandam non opus eſt hic memorare, ſtudio felici Mathematicas Diſciplinas promoverunt; univerſi tamen ante ipſum *Neutonum* vel ſpecialibus tantum curvis operam dederunt, vel neceſſariam qua uſi ſumus curvarum diſtributionem perfunctoria ſolummodo meditatione dignati ſunt. Methodus vero D. *Neutoni* neque ad omnes Lineas cujuſvis ordinis, nec expreſſe ad aliquas omnium ordinum ab ipſo Auctore extenditur; proin deſiderari videtur methodus aliqua generalis cujus ope Lineæ Geometricæ, ſecundum interſectiones ſuas cum recta in ordines diſtributæ, poſſunt ita deſcribi ut unicuique ordini ſua deſcriptio competat, non minus quam æquatio ſua generalis Algebraica; & poſt tanta nova de curvarum quadraturis inventa, quibus Geometria ampliſſime ditata eſt, pauca quidem alia ad eam Geometriæ ſublimioris perfectionem quam ſperare licet obtinendam deſiderari videntur. Nos vero tantum finem in præcedentibus aſſecutos eſſe nequaquam profitemur; cum varia de curvarum deſcriptione problemata proponi poſſint quibus conſtruendis ea non ſufficiunt; ſed ad eam Geometriæ perfectionem obtinendam viam ſtruunt quæ jam demonſtravimus, & ad eundem finem conducere poterint quæ ſequenti parte ſumus demonſtraturi.

Oſtendimus ſatis prolixe quouſque lineæ rectæ conducere poſſunt ad Lineas omnis ordinis deſcribendas; ſed ut Theoria noſtra reddatur adhuc univerſalior proxime inveſtigabimus quæ Lineæ ex aliis quibuſcunque deſcribi poterint.

PARS

Pars Secunda.

Ubi Lineæ omnium Ordinum altiorum deſcribuntur ope Linearum Ordinis cujuſcunque inferioris.

Sectio I.

Ubi demonſtratur Organica Curvarum deſcriptio Neutoniana.

Prop. I.

Circa data puncta C, S *ut Polos* (Fig. 1.) *moveantur anguli dati* FCO, KSH, *ducatur concurſus crurum* CF, SK *per Sectionem Conicam* AQS *per Polum alterutrum* S *tranſeuntem, & duo reliqua crura* CO, SH *concurſu ſuo* P *deſcribent Lineam tertii Ordinis.*

IN plano Sectionis Conicæ ducatur recta quævis ED, & inveſtigandum ſit quoties recta hæc Lineæ deſcriptæ poteſt occurrere. Supponamus crurum CO, SH concurſum P duci per rectam eam indefinitam ED, & per *Prop.* 1. *Partis* 1. concurſus crurum CF, SK deſcribet Sectionem conicam tranſeuntem per polos C, S quæ occurrere poteſt ſectioni datæ AQS in tribus punctis M, N, T.

M, N, T præter punctum S, at non in pluribus: Proinde in deſcriptione hujus *Prop.* quoties Q concurſus crurum CF, SK ſectionem AQS percurrens pervenerit ad puncta tria M, N, T, erit punctum P concurſus crurum CO, SH in recta ED, & in nullo alio caſu eam rectam tanget. Nam cum Q concurſus crurum CF, SK accedit ad punctum S recta SK ſectionem conicam tangit in S, & ſi tunc concurrant CO, SH in puncto G, id punctum nequit eſſe in recta ED niſi punctorum M, N, T aliquod coincidat cum S & duæ ſectiones CMS, AQS ſe mutuo contingant in S. Unde manifeſtum eſt curvam delineatam motu puncti P ter poſſe occurrere rectæ ED ſed non ſæpius; cumque idem de alia quavis recta in plano curvæ ducta demonſtrari poſſit, liquet Lineam deſcribi Ordinis tertii.

Hæc Propoſitio aliter demonſtratur ex *Prop.* 11. *Partis* 1. oſtendimus in ea Propoſitione ſi angulus SN*q* (*Fig.* 2.) angulari puncto N percurrat rectam datam BN, interea moveatur angulus FC*q* circa polum C, concurſus crurum C*q* & N*q* ducatur per rectam datam B*q* & denique ſimul coincidant CF & SN cum CS, punctum Q deſcripturum ſectionem conicam tranſeuntem per S non vero per punctum C. Sit hæc ſectio conica AQS, & deſcriptio hujus *Prop.* ad eam reduci poterit *Corol.* 3. *Prop.* 10. *Partis* 1. ubi curva deſcribitur concurſu cruris CO cum SR angulum datum RSN conſtituente cum SN. Proinde curva erit Linea Ordinis tertii, & deſcriptio hujus Propoſitionis ad deſcriptionem *Prop.* 10. *Partis* 1. reduci poterit, ſi ex *Prop.* 11. omnes ſectiones conicæ conſtrui poterint, & punctum S in ſectionis puncto quovis exiſtere.

Corol. I. Curva habet punctum duplex in S, & tranſit ſemel per C; ut ſatis manifeſtum eſt ex *Corol.* 1. *Prop.* 10. *Partis* 1. ſi AQS ſit ſectio conica earum quæ deſcribi poſſunt methodo *Prop.* 11. Sed idem hinc facile conſtat univerſaliter quod curva tranſeat per S (*Fig.* 1.) cum CO coincidit cum CS, & (ſi ICS = FCO) CF cum CI vel CU, & SK cum SI vel SU; proinde curva tranſit per S in duabus Poſitionibus rectæ SH cujus concurſu cum CO curvæ puncta determinantur, adeoque erit S punctum curvæ duplex, & SH in duabus iis Poſitionibus duos arcus curvæ concurrentes in S continget.

Corol. II. Hinc ſi recta CI angulum ICS conſtituens æqualem dato FCO bis occurrat ſectioni conicæ, curva habebit *Nodum* vel *Crucem* in S. Quod ſi IC ſectionem contingat, curva habebit *Cuſpidem* in S; ſi denique recta IC tota ſit extra ſectionem conicam, curva habebit *Punctum* conjugatum in S: In primo caſu duo curvæ arcus tranſeunt per C, in ſecundo punctum P ſemel ad punctum S accedendo & ab eodem puncto dein regrediendo duos deſcribit arcus qui cuſpidem acutiſſimum formant in S; in tertio vero caſu recta CO nunquam coincidit cum CS.

Corol. III.

Corol. III. Curva abit in infinitum (*Fig.* 1.) quoties crura CO, SH evadunt parallela, ſive quoties angulus CQS evadit ſupplementum angulorum QCO & QSH ad quatuor rectos; proinde ſi ſuper CS deſcribatur arcus circuli, cui angulus æqualis dicto ſupplemento inſcribi poterit, is in tribus punctis præter polum S occurret Sectioni conicæ AQS; cumque punctum Q concurſus crurum CF, SK ad hæc tria puncta pervenerit curva abibit in infinitum, ſint ea puncta *m*, *n*, *t* & ad rectas C*m*, C*n*, C*t* conſtituantur anguli XC*m*, YC*n*, ZC*t* æquales dato FCO, & rectæ CX, CY, CZ erunt parallelæ tribus curvæ Aſſymptotis & plagas monſtrant crurum infinitorum.

Corol. IV. Ducatur tangens ſectionis conicæ ad punctum *m* (*Fig.* 1.) & in tangentem ex polis C, S demittantur perpendiculares P & *p*, ſumatur dein CV ad CS, ut $P \times Sm^2$ ad $P \times Sm^2 - p \times Cm^2$, & ſi ad V conſtituatur recta parallela rectæ CX, ea erit Aſſymptotos curvæ, & ſimili ratione aliæ duæ Aſſymptoti determinari poſſunt. Quod ſi circulus cui inſcriptus eſt angulus æqualis ſupplemento datarum QCO, QSH ad quatuor rectos ſectionem conicam AQS contingat in *m* coibunt puncta duo *m*, *n* vel *m*, *t* & duæ Aſſymptoti parallelæ rectis CX, CY vel CX, CZ coeuntes ſimul abibunt in infinitum, & crura duo curvæ erunt Parabolica. Quippe ſi recta ſectionem conicam contingens etiam contingat circulum, erit ex natura circuli $P : p :: Cm^2 : Sm^2$, adeoque cum $CV : CS :: P \times Sm^2 : P \times Sm^2 - p \times Cm^2$ erit CV infinita.

Corol. V. Ex præcedentibus *Corollariis* curvæ ſpecies non difficulter indagari poteſt; quippe ducta recta CI ſecundum *Corollarium* 2. facile conſtabit an curva ſit *Nodata*, *Cuſpidata* an *Punctata* in S; & ducto circulo ſecundum *Corollarium* 3. ex ejus interſectionibus cum ſectione conica AQS innoteſcet numerus & poſitio crurum curvæ infinitorum. Si circulus in tribus punctis præter polum S ſectioni conicæ occurrat, curva conſtabit ex tribus Hyperbolis ad tres Aſſymptotos; ſi circulus ſectioni occurrat in punctis duobus S, T, & in alio puncto ſectionem contingat, curva duo habebit crura Parabolica & totidem Hyperbolica; ſi circulus ſectioni occurrat in uno tantum puncto præter S, curva unicam habebit Aſſymptoton eritque Hyperbola defectiva; ſi circulus ſectionem contingat tantum nec eum omnino ſecet curva erit Parabolica. Si ſectio conica AQS ſit Parabola vel Hyperbola, & S ſit terminus cruris alicujus infiniti, & loco anguli KSH ſubſtituatur recta quæ ſemper movetur parallela alicui ſectionis conicæ Aſſymptoto, curva habebit punctum duplex ad diſtantiam infinitam, eritque Hyperboliſmus Hyperbolæ, Hyperboliſmus Parabolæ, Parabola Carteſiana vel Parabola Cubica.

Corol. VI. Si (*Fig.* 3.) crura CO, SH ſimul coincidant cum CS, curva hac Propoſitione deſcripta erit Sectio Conica; ſit D punctum ubi recta ED occurrit datæ CS, & ſit B punctum ad quod cum accedit Q coincidunt

incidunt ſimul CO, SH cum CS, & ſi punctum P cujus motu curva deſcribitur ducatur per rectam ED, cum P pervenerit ad D concurſus crurum CF, SK erit in B; proinde ſectio conica deſcripta concurſu crurum CF, SK occurret ſectioni AQS in punctis B & S & præterea in aliis duobus M, N, cumque in deſcriptione hujus Propoſitionis pervenerit Q ad B, punctum P non erit in recta ED, niſi M vel N coincidat cum B; proinde curva deſcripta occurret rectæ ED tunc tantum cum Q pervenerit ad M vel N, adeoque bis tantum ſecabit rectam ED ſeu aliam quamvis in plano curvæ ductam, & proinde Linea deſcripta erit ſecundi Ordinis ſeu ſectio conica. In hoc caſu punctum S evadit ſimplex, & Linea non tranſit per punctum C.

Corol. VII. Sit ſectio conica AQS circulus; cumque in duobus tantum punctis circuli ſibi mutuo occurrere poſſint, manifeſtum eſt ex *Corollario* 5. Hyperbolam deſcribi defectivam quæ unicam habet Aſſymptoton. Quod quidem prius demonſtravimus in *Corollariis* 3 & 4, *Lemmatis* 2, *Partis* 1, ubi anguli FCO, KSH ſibi mutuo ſupplementa ſunt ad duos rectos.

Corol. VIII. Si AQS *(Fig.* 1.*)* ſit Parabola Apollonia, S ejus vertex punctum vero C ſit ubicunque in ejus axe, anguli KSH, FCO recti, Lineæ erunt eæ ipſæ quas *Exemplo* 22. *Schol. Prop.* 9. *Partis* 1. deſcribere docuimus ſpecierum 68, 69 vel 70. Si S ſit in vertice Hyperbolæ, æquilateræ *(Fig.* 4.*)* punctum C ſit in axe figuræ, ſit CS $= b$, PM $= y$, SM $= x$, & latus tranſverſum dicatur a erit $y^2x - \overline{b-a} \times y^2 = x^3 - bx^2$, quæ nullo labore ad formam *Neutonianam* reduci poteſt; atque curva erit ſpeciei 11, 12 vel 13. Si S ſit in concurſu *(Fig.* 5.*)* Hyperbolæ æquilateræ cum Aſſymptoto ſua ad diſtantiam infinitam, ſitque C ubicunque in eadem Aſſymptoto angulus FCO rectus, & recta SH moveatur ſemper parallela Aſſymptoto CS occurrens cruri CF in Q puncto Hyperbolæ, Linea deſcripta concurſu rectæ SH cum CO erit Parabola Carteſiana cujus æquatio ad Aſſymptoton CS (ſi diſtantia puncti C a centro Hyperbolæ dicatur b, & ordinata Hyperbolæ ad punctum C Aſſymptoto perpendicularis dicatur a) $byx - abx = y^3$. Si cæteris manentibus ſit S in concurſu Hyperbolæ cum altera Aſſymptoto in quo non eſt C curvæ æquatio erit $aby = -x^3 + bx^2$, adeoque curva deſcripta erit Parabola Cubica omnium fere Linearum tertii Ordinis maxime celebris. Harum inveſtigatio cuivis qui calculum tentaverit erit facillima; plura vero hujuſmodi exempla proferre ſupervacaneum duco.

Corol. IX. Hujus Propoſitionis ope Linea tertii Ordinis duci poteſt per ſeptem data puncta quorum unum eſt punctum duplex. Dentur *(Fig.* 6.*)* ſeptem puncta S, C, M, N, T, R, L, Lineæ deſcribendæ quarum S duplex eſt. Jungantur punctum S & alia duo quævis puncta ut C, M, & trianguli MCS rotetur tum angulus MSC circa polum S, tum

tum angulorum reliquorum alteruter MCS circa verticem ſuum C; applicetur cóncurſus crurum CM, SM ſucceſſive punctis datis N, T, R, L, & interea concurrant crura reliqua CS & SC in punctis quatuor B, D, E, F. Per puncta illa quatuor & quintum S ducatur ſectio conica per *Prop.* 4. *Partis* 1. & anguli MSC, MCS ita rotentur ut crurum SC, CS concurſus percurrat ſectionem illam conicam & concurſus reliquorum crurum deſcribet Curvam per data puncta S, C, M, N, T, R, L tranſeuntem quæ punctum duplex habebit in Polo S per *Corol.* 1. hujus Propoſitionis. Lineas vero tertii Ordinis quæ punctum habent duplex ex ſex ſimplicibus & puncto duplice determinari hinc patet; quod ſi duæ Lineæ tertii Ordinis per eadem ſex puncta tranſirent, & in eodem ſeptimo punctum duplex haberent, concurſus Linearum duarum tertii Ordinis eſſent decem, cum plures novem eſſe nequeant.

Prop. II.

Cæteris manentibus ut in Prop. 1. *ſi ſectio Conica quam percurrit punctum* Q *concurſus crurum* CF, SK (Fig. 7.) *tranſeat per neutrum Polorum* S, C, *punctum* P *concurſus crurum* CO *&* SH *deſcribet Lineam quarti Ordinis.*

IN plano ſectionis conicæ ducatur ut in demonſtratione *Prop.* præcedentis recta quævis ED, & inveſtigandum ſit quoties recta hæc Lineæ deſcriptæ occurrere poſſit. Supponamus ut in Propoſitione præcedente crurum CO, SH concurſum P duci per rectam ED, & per *Prop.* 1. *Partis* 1. concurſus crurum CF, SK interea deſcribet ſectionem coni tranſeuntem per polos C, S, quæ occurrere poteſt datæ ſectioni AQB in punctis quatuor M, N, T, R; cum ex Hypotheſi Poli C, S non ſint in ſectione. Proinde in deſcriptione hujus Propoſitionis cum Q pervenerit ad puncta M, N, T, R erit P concurſus crurum CO & SH, cujus motu curva deſcribitur, in recta ED, & in nullo alio caſu rectam ED contingere poteſt; unde manifeſtum eſt curvam motu puncti P delineatam quater poſſe occurrere rectæ ED & non ſæpius; cumque

idem de alia quavis recta in plano ducta demonſtrari poſſit, patet Lineam deſcribi quarti Ordinis.

Hæc Propoſitio eadem ratione ex *Corollario* ultimo *Prop.* 15. demonſtrari poſſet qua Propoſitionem præcedentem ex *Prop.* 11. Partis primæ demonſtravimus.

Corol. I. Curva hujus Propoſitionis tria habet puncta duplicia. Imprimis punctum C erit duplex; quippe ſi angulus ISC ſit æqualis dato KSH, & recta SI producta occurrat ſectioni conicæ in punctis I, U, cum Q pervenerit ad puncta I, U curva tranſibit per C & recta CO tangens erit ad duos curvæ arcus in C concurrentes; nam in utroque caſu coincidet SH cum SC, adeoque concurſus crurum CO, SH erit in ipſo puncto C. Similiter oſtendi poteſt punctum S eſſe duplex. Adhæc ſi producatur recta CS donec ſectioni occurrat in punctis duobus A, G & punctum Q accedat ad A, & dein ad G, erit concurſus crurum CO, SH in eodem puncto B (vid. *Corol.* 8. *Prop.* 2. *Partis* 1.) in utroque caſu, adeoque id punctum B erit duplex. Puncta igitur C, S, B ſunt omnia duplicia.

Corol. II. Sit angulus SCZ = FCO, & curva erit *Nodata*, *Cuſpidata* aut *Punctata* in punctis C, S, B prout rectæ SI, CZ, CS productæ ſectionem AQB ſecant, contingant aut totæ extra eam locantur. Hæ igitur curvæ in ſpecies decem inſigniores diſtingui poſſunt abſque ullo reſpectu habito ad crurum vel diametrorum numerum aut poſitionem; aliæ enim tres habent nodos, aliæ tres cuſpides aliæ tria puncta conjugata; aliæ habent duos Nodos & unicam Cuſpidem, aliæ duos Nodos & unicum Punctum conjugatum, aliæ duas Cuſpides & Nodum unicum, aliæ duo Puncta conjugata & Nodum unicum; aliæ habent Nodum, Cuſpidem & Punctum conjugatum, aliæ duas Cuſpides & unicum Punctum conjugatum, & aliæ denique unicam Cuſpidem & duo Puncta conjugata. Nodi vero nonnunquam formam crucis induunt & alia plurima harum curvarum oriuntur diſcrimina quibus enumerandis non licet immorari.

Corol. III. Curva abit in infinitum quoties crura CO, SH evadunt parallela, & angulus CQS æqualis ſupplemento datorum angulorum FCO, KSH ad quatuor rectos. Super CS deſcribatur circulus (*Fig.* 8.) habens angulum dicto ſupplemento æqualem ei inſcriptum occurrens ſectioni Conicæ in quatuor punctis M, N, T, R; & ubi accedit Q ad ea puncta crura CO, SH evadunt parallela, & curva abit in infinitum. Proinde Aſſymptoti harum curvarum poſſunt eſſe quatuor, duæ vel nullæ, cum circulus is ſectioni occurrere poſſit in punctis quatuor, duobus vel nullis. Si circulus ſectionem contingat, crura erunt Parabolica; & univerſaliter curvæ hac Propoſitione deſcriptæ vel octo vel quatuor vel nulla habebunt crura Hyperbolica, vel quatuor habebunt Parabolica vel duo, vel quatuor Hyperbolica & ſimul duo Parabolica. Atque ex hoc & præcedenti

Corollario

Corollario Lineæ hac Propoſitione deſcriptæ in 60 ſpecies generales dividi poſſunt.

Corol. IV. Si crura CO, SH ſimul coincidant cum CS Linea deſcripta erit tertii Ordinis, & puncta C, S erunt ſimplicia; ſit Δ punctum ad quod cum Q perveneri coincidunt CO, SH cum CS, *(Fig. 8.)* & Δ erit in concurſu ſectionis Coni deſcriptæ concurſu rectarum CF, SK dum P ducitur per rectam ED; nam ubi P accedit ad D, rectæ CF, SK, concurrunt in Δ; ſed in deſcriptione Propoſitionis hujus cum Q pervenerit ad Δ, punctum P non erit in recta ED, niſi duo concurſus ſectionum Conicarum ſint in Δ; unde manifeſtum eſt punctum P ter tantum exiſtere poſſe in recta quavis ED, adeoque Lineam deſcribi tertii Ordinis. In hoc caſu curva unicum habet punctum duplex in B.

Prop. III.

Cæteris manentibus ut in Propoſitionibus præcedentibus ducatur concurſus crurum CF, SK *per Lineam quamcunque* AB (Fig. 9.) *Ordinis (n) & crurum reliquorum* CO, SH *concurſus deſcribet Lineam Ordinis 2n.*

IN plano curvæ ducatur recta quævis ED & inveſtigandum ſit in quot punctis hæc recta Lineæ Propoſitione deſcriptæ occurrat. Ducatur punctum P concurſus crurum CO, SH per rectam indefinitam ED, & concurſus crurum CF, SK interea deſcribet Lineam Ordinis ſecundi per *Prop.* 1. *Partis* 1. Proinde in deſcriptione hujus Propoſitionis punctum P tum tantum erit in recta ED cum punctum Q concurſus crurum CF, SK ſit in ea ſectione Coni, ſed punctum Q ducitur per Lineam ordinis *n.* ex hyp. quæ occurrere poteſt ſectioni Conicæ in punctis 2*n* ex natura Linearum; & ubi Q puncta hæc 2*n* tangit punctum P reperitur in recta ED; unde manifeſtum eſt Lineam motu puncti P deſcriptam rectæ ED occurrere in punctis 2*n* adeoque eam Lineam eſſe ordinis 2*n*.

Corol. I. Curvæ arcus *(n)* concurrunt in puncto C & totidem in puncto S; quippe ſi angulus CSI ſit æqualis dato HSK recta SI poterit occurrere Lineæ datæ AB in punctis *n*; ſed cum Q ad quodvis horum punctorum pervenerit in deſcriptione Propoſitionis coincidet SH cum SC, adeoque curva tranſibit per C; proinde arcus curvæ *(n)* concurrunt

runt in Polo C, & eadem ratione oſtendi poteſt totidem arcus concurrere in Polo S. Sed aliud etiam eſt tertium punctum curvæ Δ ubi concurrunt crura CO, SH cum CQ, SQ ſimul coincidunt cum CS ubi arcus curvæ *(n)* concurrunt; quoniam recta CS in punctis *n* occurrere poteſt Lineæ AB. Cujus vero generis puncta ſint C, S, Δ innoteſcet methodo ſimili ei quam explicuimus *Corol.* 2. *Prop.* præcedentis.

Corol. II. Lineæ methodo hujus Propoſitionis deſcriptæ eæ ſunt quæ plura habent puncta in quibus arcus *n* concurrunt quam aliæ quævis ejuſdem ordinis. Quippe Linea ordinis $2n$ nequit quatuor habere nodos arcuum curvæ *(n)* in eodem puncto concurrentium; cum in ea Hypotheſi ſectio Coni per ea quatuor puncta multiplicia & quintum ſimplex ducta curvæ occurreret in punctis $4n+1$, quod eſt impoſſibile; ſectio etenim Conica Lineam ordinis $2n$ ſecare poteſt in punctis tantum $4n$.

Corol. III. Aſſymptoti harum curvarum generaliter determinantur eadem methodo qua eas in caſibus particularibus *Prop.* 1 & 2 prius inveſtigavimus, ducendo ſcil. circulum ſuper CS, cui inſcribitur ſupplementum angulorum FCO, KSH ad quatuor rectos, & hujus circuli occurſus cum Linea data AB notando; nam cum punctum Q ad dictos occurſus pervenerit curva abibit in infinitum.

Corol. IV. Hæc omnia obtinent ex Hypotheſi quod puncta C, S non ſint in curva AB. Sed ſi alterum punctorum C, S ſit in curva data AB, Linea motu puncti P deſcripta erit ordinis $2n-1$. Quod ſi curva AB punctum habeat in quo arcus curvæ concurrunt quorum numerus eſt *r*, & Polus C ſit in eo puncto, Linea deſcripta motu puncti P erit ordinis $2n-r$. Hæc ex Propoſitionis hujus demonſtratione facile intelligi poſſunt.

Prop. I, & II. Poſt enumerationem Linearum tertii Ordinis protulit *Neutonus*, atque eas univerſaliores reddi poſſe ibidem innuit; quod *Prop.* hac 3 conati ſumus efficere. In his unicam Lineam curvam & duos tantum angulos ad Lineas ſuperiores ducendas adhibuimus; ſequentibus vero omnes *Partis* 1. Propoſitiones univerſales reddere conabimur, quemadmodum hanc Propoſitionem 1. *Partis* 1. quam fieri poteſt generalem reddidimus.

SECTIO

Sectio II.

Ubi Curvæ inveſtigantur quæ ex aliis quibuſcunque angulorum datorum motu deſcribi poſſunt.

Prop. IV.

Cæteris manentibus ut in Prop. 5. Partis 1. *loco rectæ* BQ (Fig. 10.) *ſubſtituatur Linea quævis* BQG *ordinis n, & ſi concurſus crurum* CF, SN *ducatur per eam curvam concurſus crurum* CO, NL *deſcribet Lineam ordinis 3n, ſi Linea* BQG *non tranſeat per Polos* C, S.

IN plano curvæ BQG ducatur recta quævis ED, & inveſtigare oportet quoties Linea deſcripta motu puncti P occurrere poterit huic rectæ. Ut id innoteſcat ſupponamus concurſum crurum CO, NL duci per rectam eam indefinitam ED & concurſus crurum CF, SN interea deſcribet Lineam tertii Ordinis per *Prop.* 10. *Partis* 1. quæ occurrere poteſt Lineæ BQG in punctis 3*n*. Unde ubi punctum P exiſtit in recta ED in deſcriptione hujus Propoſitionis erit punctum Q concurſus crurum CF, SN ſimul in Linea dicta tertii Ordinis & in Linea BQG, & viciſſim quoties punctum Q accedit ad concurſum quemvis Lineæ dictæ tertii Ordinis & Lineæ BQG erit punctum P in recta ED, ii vero concurſus ſunt 3*n*, adeoque Linea motu puncti P deſcripta ſecare poteſt rectam ED in punctis 3*n* & nunquam in pluribus, cumque idem de recta quavis alia demonſtrari poſſit, manifeſtum eſt Lineam deſcribi concurſu crurum CO, NL ordinis 3*n*.

Corol. I.

Corol. I. Curvæ arcus $2n$ in polo C concurrunt; quippe ſi ſuper CS deſcribatur circuli arcus cui inſcribitur angulus datus SNL occurrens rectæ AN in punctis M & m, & jungantur SM, Sm ſingulæ occurrentes curvæ BQG in punctis n, manifeſtum eſt quoties punctum Q pervenerit ad occurſus hos rectarum SM, Sm cum curva BQG crus NL tranſiturum per C, adeoque curvam per idem punctum tranſituram; ſunt igitur puncta $2n$ curvæ BQG ad quæ cum punctum Q pervenerit curva tranſibit per C, & proinde arcus curvæ $(2n)$ in eo puncto concurrunt. Curva vero per polum S tranſire nequit cum recta NL, in qua ſemper verſatur P punctum curvæ, per S nunquam poſſit tranſire.

Corol. II. Si circulus ſuper CS deſcriptus cui inſcribitur angulus SNL rectam AN contingat curva habebit cuſpides in puncto C quarum numerus erit n, ſi AN tota ponatur extra circulum erit C punctum curvæ conjugatum. Punctum C erit partim *Nodata* partim *Cuſpidata* ſi recta SM occurrat Lineæ BQG ſed Sm eam contingat.

Corol. III. Super CS deſcribatur arcus circuli cui inſcribi poſſit angulus æqualis ſupplemento angulorum SNL, FCO ad duos aut quatuor rectos occurrens Lineæ BQG in punctis $2n$; & cum punctum Q ad occurſum eorum quemvis pervenerit curva abibit in infinitum, & crus CO erit parallelum Aſſymptoto curvæ, atque hinc Plaga Aſſymptoton quarum numerus eſt $2n$ determinari poteſt. Sed cum punctum N abit in infinitum ſeu SN evadit parallela rectæ AN curva abit in infinitum; cumque recta SN occurrere poſſit Lineæ BQG in punctis n; hinc plagæ aliarum Aſſymptoton quarum numerus eſt (n) determinari poſſunt; atque hæ cum prioribus Aſſymptotos conſtituunt $3n$ & plures Lineas ordinis $3n$ nequit habere.

Corol. IV. Si Polus C ſit in BQG curva hac Propoſitione deſcripta erit Linea ordinis $3n - 1$; ſi Polus S ſit in Linea BQG deſcribetur Linea ordinis $3n - 2$; & ſi utrumque contingat deſcribetur Linea ordinis $3 \times \overline{n - 1}$.

Corol. V. Si in quadrilatero CQNP trium punctorum P, N, Q alterum ducatur per rectam indefinitam, alterum per Lineam ordinis n, tertium deſcribet Lineam ordinis $3n$: Quippe ſi punctum illud tertium cujus motu curva deſcribitur ſupponatur duci per rectam quamcunque punctum quod duci ſuppoſuimus per Lineam ordinis (n) deſcribet Lineam Ordinis tertii per *Prop.* 5, 10 & 13. *Partis* 1. quæ occurret Lineæ ordinis (n) in punctis $3n$; atque dum in iis occurſibus $3n$ verſatur, punctum cujus motu curva deſcribitur erit in locis $3n$ ejuſdem rectæ. Unde generaliter conſtat ſi illorum trium punctorum aliud percurrat rectam, aliud Lineam ordinis (n) tertium deſcripturum Lineam ordinis $3n$.

Corol. VI.

Corol. VI. Si BQG ſit ſectio Conica, Lineæ hac *Prop.* deſcriptæ erunt ſexti Ordinis ſi ſectio non tranſeat per Polos C, S; ſed ſi Polus C ſit in ſectione Linea deſcripta erit Ordinis quinti, ſi Polus S ſit in ſectione, Linea erit Ordinis quarti, & ſi uterque ſit in ſectione Linea erit Ordinis tertii & deſcriptio coincidet cum iis *Prop.* 5 & 13. *Partis* 1.

PROP. V.

Ducatur concurſus crurum CF, SN *ut in* Prop. *præcedenti* (Fig. 11.) *per Lineam Ordinis (n) & ſi punctum angulare* N *ſimul ducatur per Lineam Ordinis m, concurſus crurum* CO, NL *deſcribet Lineam Ornis 3nm.*

SIT BQG Linea ordinis *n*, BNA Linea ordinis *m*, & in plano curvarum ducatur recta infinita quævis ED, inveſtigare oportet quoties Linea deſctipta motu puncti P huic rectæ occurrat. Ducatur punctum P concurſus crurum CO, NL per rectam ED, & interea percurrat punctum N Lineam BNA ordinis *m*, atque per *Corol.* 5. *Prop.* præcedentis punctum Q deſcribet Lineam ordinis 3*m*, quæ occurret Lineæ BQG ordinis *n* in punctis 3*mn*: Sed in deſcriptione hujus Propoſitionis quoties punctum Q (quod percurrit Lineam BQG) accedit ad concurſum quemvis curvarum illarum, punctum P eſt in recta ED. Sunt igitur puncta 3*nm* in curva BQG, ad quæ ubi accedit Q, punctum P tangit rectam ED; ptoinde Linea motu puncti P deſcripta ſecabit rectam ED in punctis 3*nm*; cumque idem de alia quavis recta in plano ducta demonſtrari poſſit, manifeſtum eſt Lineam motu puncti P deſcriptam eſſe ordinis 3*nm*.

Corol. I. Curvæ tot arcus concurrunt in C quot ſunt unitates in numero 2*nm*. Quippe ſi ſuper CS deſcribatur arcus circuli cui inſcribitur angulus SNL, is occurret Lineæ BNA in punctis 2*m*; & ſi ad hos occurſus ducantur rectæ ex Polo S, eæ occurrent Lineæ BQG in punctis 2*mn*; ſed in deſcriptione hujus Propoſitionis cum punctum Q (quod percurrit Lineam BQG) ad occurſuum illorum aliquem pervenerit, recta NL tranſibit per C, adeoque coincidet P cum C; & proinde curvæ arcus (2*mn*) concurrunt in eodem puncto C, & punctum illud erit ſemper multiplex ſi *m* vel *n* ſuperent unitatem.

N *Corol.* II.

Corol. II. Deſcribatur ſuper CS arcus circuli cui inſcribitur angulus æqualis ſupplemento datorum FCO, SNL ad quatuor rectos occurrens Lineæ BQG in punctis $2n$, cumque punctum Q ad aliquem occurſuum illorum pervenerit evaneſcet angulus CPN, adeoque CO, NL erunt parallelæ, & curva abibit in infinitum. Alia formantur curvæ crura cum N abit in infinitum, & SN evadit parallela cruri infinito Lineæ BNA ordinis m. Curvarum hac Propoſitione deſcriptarum Aſſymptoti ſunt (n) quarum ſingulæ tot alias habebunt ſibi parallelas quot unitates ſunt in numero m, ſi circulus cui inſcribitur ſupplementum angulorum FCO, SNL ad quatuor rectos Lineæ BQG occurrat in punctis $2n$, & ſingulæ rectæ ex S eductæ ad occurſus illos ſecent Lineam BNA in punctis m.

Corol. III. Eadem ratione qua Propoſitionem demonſtravimus oſtendi poteſt univerſaliter ſi in quadrilatero CPNQ trium punctorum P, Q, N alterum ducatur per Lineam ordinis (n) alterum per Lineam ordinis (m) tertium deſcripturum Lineam ordinis $3nm$.

Prop. VI.

Cæteris manentibus ut in Prop. 14. Partis 1. (Fig. 12.) *loco rectarum* AN, BQ, DR *ſubſtituantur Lineæ quævis ordinum m, r, n & Linea motu puncti* P, *concurſus crurum* CN & SR, *deſcripta erit ordinis* $4mnr$.

SUpponamus imprimis puncta N & Q rectas infinitas AN, BQ percurrere, ſed punctum R duci per Lineam ordinis n; & in hoc caſu punctum P deſcribet Lineam ordinis $4n$. Quippe ſi punctum P per rectam quamvis ED ducatur punctum R deſcribet Lineam ordinis quarti per *Prop.* 19. *Partis* 1. quæ occurret Lineæ ordinis n in punctis $4n$ cumque R pervenerit ad occurſus illos $4n$ erit P in recta ED, adeoque punctum P deſcribet Lineam ordinis $4n$. Eadem ratione oſtendi poteſt univerſaliter ſi in quadrilatero PNQR duo quævis puncta ducantur per rectas poſitione datas & tertium percurrat Lineam ordinis n quartum deſcripturum Lineam ordinis $4n$.

Supponamus ſecundo puncta duo R, N duci per Lineas ordinum n & punctum Q per rectam BQ; & in hoc caſu punctum P deſcri-

Lin-

Lineam ordinis $4nm$: Quippe ſi punctum P ducatur per rectam ED, & punctum N per Lineam ordinis m, punctum R deſcribet per modo demonſtrata Lineam ordinis $4n$ quæ occurret Lineæ DR in punctis $4nm$; cumque R ad occurſus illos $4nm$ pervenerit erit P in recta ED, adeoque Linea motu puncti P deſcripta erit ordinis $4nm$; & in genere ſi punctorum P, Q, N, R duo quævis ducatur per Lineas ordinum m, n, tertium vero percurrat rectam, quartum deſeribet Lineam ordinis $4nm$.

Supponamus denique ut in Propoſitione puncta tria N, Q, R duci per Lineas ordinum m, r, n; & punctum P deſcribet Lineam ordinis $4nmr$. Quippe ſi N & Q ducantur per Lineas ordinum m, r, & P percurrat rectam ED, punctum R deſcribet Lineam ordinis $4mr$ quæ occurret Lineæ DR ordinis (n) in punctis $4mnr$; ſed ubi R pervenerit ad occurſum quemvis illarum Linearum in deſcriptione hujus Propoſitionis, erit punctum P in recta ED: Proinde Linea motu puncti P deſcripta occurrit rectæ ED in punctis $4nmr$; cumque idem de alia quavis recta demonſtrari poſſit manifeſtum eſt Lineam motu puncti P deſcriptam fore ordinis $4nmr$.

Corol. I. Curvæ tot arcus concurrunt in C quot ſunt unitates in numero $2nmr$; & totidem concurrunt in altero Polo S; coincidat enim SR cum SC (*Fig.* 13.) & tot poſſunt eſſe variæ poſitiones rectæ RQ quot ſunt unitates in numero (n) quoniam in tot punctis recta SC occurrere poteſt Lineæ ordinis n. Sed pro ſingulis poſitionibus puncti R poſitiones puncti Q poſſunt eſſe tot quot ſunt unitates in numero (r) & pro ſingulis poſitionibus puncti Q tot poſſunt eſſe poſitiones puncti N quot ſunt unitates in numero $2m$, ſiquidem concurſus circuli ſuper CSQ deſcripti, cui inſcribitur angulus CNQ, cum Linea AN denotant puncta N; proinde ubi coincidit SR cum SC poſitiones rectæ CN poſſunt eſſe $2nrm$, ſed Linea Propoſitione hac deſcripta tranſit per C ubi coincidit SR cum SC; adeoque arcus $2nrm$ curvæ concurrunt in Polo C; eadem ratione demonſtratur Lineæ puncta $2nmr$ coire in altero Polo S. Cum vero Linea deſcripta ſit ordinis $4nmr$, patet rectam CS in nullo alio puncto præter Polos C, S occurrere Lineæ in Propoſitione deſcriptæ.

Corol. II. Si $n = m = r = 2$ ſive curvæ omnes AN, BQ, DR ſint ſectiones conicæ Linea deſcripta erit Ordinis 32: ſi AN, BQ, DR ſint omnes Lineæ tertii Ordinis Linea deſcripta motu puncti P erit ordinis 108.

PROP. VII.

Sint cætera omnia ut in Prop. 21. Partis 1. *ſed pro rectis* BQ, AN, DR, GL, *&c. ſubſtituantur Lineæ curvæ ordinum* $m, r, s, t,$ *&c. ſitque n numerus curvarum omnium, & Linea deſcripta motu puncti* P, *concurſus crurum* CO, LT (Fig. 14.) *erit ordinis* $mrst \times \overline{n+1}$.

SUpponamus imprimis puncta omnia Q, N, R, L recta ſuas percurrere ut in *Prop.* 21. præter ipſum Q, & ſi punctum Q ducatur per Lineam ordinis m punctum P deſcribet Lineam ordinis $mn+m$. Quippe ſi punctum P ducatur per rectam quamvis ED & puncta L, R, N rectas ſuas percurrant, punctum Q deſcribet Lineam ordinis $n+1$ per *Prop.* 21. *Partis* 1. quæ occurret Lineæ BQ ordinis m in punctis $mn+m$. Cumque punctum Q dum percurrit Lineam BQ ad occurſus illos $mn+m$ pervenerit erit punctum P in recta ED, adeoque Linea motu puncti P deſcripta erit ordinis $mn+m$. Simili ratione ex *Prop.* 22. oſtendi poteſt univerſaliter ſi omnia puncta P, Q, N, R, L, &c. ducantur per rectas poſitione datas præter duo quævis & horum alterum ducatur per Lineam ordinis (m) alterum deſcripturum Lineam ordinis $mn+m$.

Supponamus ſecundo puncta duo N, Q deſcribere Lineas ordinum r, m, & punctum P deſcribet Lineam ordinis $mrn+mr$. Quippe ſi P ducatur per rectam ED, punctum N per Lineam ordinis (r) reliqua vero præter Q per rectas poſitione datas, punctum Q deſcribet per modo demonſtrata Lineam ordinis $rn+r$ quæ occurret Lineæ ordinis m in punctis $mrn+mr$; cumque punctum Q ad occurſus illos pervenerit erit punctum P in recta ED, adeoque Linea motu puncti P deſcripta erit ordinis $mrn+mr$. Et ſimili ratione univerſaliter demonſtratur ſi omnia puncta P, Q, N, R, L, &c. præter tria quævis ducantur per rectas poſitione datas & trium reliquorum duo quævis ducantur per Lineas ordinum m, r, tertium deſcripturum Lineam ordinis $mrn+mr$.

Supponamus tertio tria puncta Q, N, R duci per Lineas ordinum m, r, s, & punctum P deſcribet Lineam ordinis $mrsn+mrs$. Quippe ſi P ducatur per rectam ED & puncta R, N per curvas ordinum s, r, punctum

punctum Q per modo demonſtrata deſcribet Lineam ordinis $srn+sr$, quæ occurret Lineæ BQ ordinis (m) in punctis $mrsn+mrs$, ad quæ cum accedit Q punctum P tangit rectam ED. Proinde puncta ſunt $mrsn+mrs$ in curva BQ in quibus ſit punctum Q & ſimul erit punctum P in recta ED; unde Linea motu puncti P deſcripta erit ordinis $mrsn+mrs$.

Similiter demonſtrari poteſt Lineam motu puncti P deſcriptam fore ordinis $mrstn+mrst$ ſeu $mrst\times\overline{n+1}$. ſi puncta omnia Q, N, R, L, &c. ducantur per Lineas ordinum m, r, s, t, &c. Et quotcunque aſſumantur curvæ earum omnium indices ductæ imprimis in ſe mutuo & dein in numerum curvarum unitate auctum oſtendunt ordinem Lineæ deſcriptæ motu puncti P concurſus crurum CO, LT.

Corol. I. Curvæ tot arcus concurrunt in Polo C quot unitates ſunt in producto $mrstn$.

Corol. II. Sit (u) numerus binarius ad poteſtatem (n) elevatus & ſi omnes curvæ datæ ſint ſectiones conicæ Linea deſcripta erit ordinis $nu+u$. Unde ſeptem ſectionum conicarum ope Lineæ deſcribi poſſunt ordinis 1024, atque decem ſectionum ope Lineæ deſcribi poſſunt ordinis ſupra 1.1000. Quæ quidem deſcriptio in ſe ſatis eſt complexa ſimpliciſſima vero eſt ſi ordo Lineæ ejuſque implicatiſſima forma conſiderentur.

PROP. VIII.

Sint reliqua omnia ut in Prop. 24. Partis 1. *ſed loco rectarum* AQ, BN, ER, FT *ſumantur Lineæ curvæ ordinum l, k, t, i & loco rectarum* DM, GE, HK *ſumantur Lineæ ordinum p, u, z, denotent n, m numeros curvarum & Linea deſcripta concurſu rectarum* CN, SM *erit ordinis* $lkitpuz\times\overline{n+m+2}$.

HÆC Propoſitio eadem ratione ex *Prop.* 24. *Partis* 1. deduci poteſt qua Propoſitionem præcedentem ex 21 *Partis* 1. demonſtravimus. Et ſimili quoque ratione oſtendi poteſt ſi in *Prop.* 26. *Partis* 1. loco rectarum ſubſtituantur curvæ eorundem ordinum Lineam deſcriptam fore ordinis $lkitpuz\times\overline{ms+nr+s+r}$.

Propoſi-

Propoſitionibus his ordinem inveſtigavimus altiſſimum Linearum quæ propoſita ratione deſcribi poſſunt; ea vero Theoremata non ſunt ita intelligenda quaſi adeo ardui generis curva in omni caſu deſcriberetur; cum ſingula methodos ſuppeditent particulares Lineas aliquas deſcribendi omnis ordinis inferioris quam in Propoſitione exprimitur.

SCHOLIUM.

Si curvæ ordinis infiniti quæ *Mechanicæ* dici ſolent in his deſcriptionibus ſubſtituantur curvæ reſultantes erunt etiam ordinis infiniti; atque hinc complexiores curvæ etiam hujus ordinis ex ſimplicioribus deſcribi poſſe videntur, & Linea methodo *Prop.* 3. hujus *Partis* ex alia quavis mechanica BQG deducta rectæ videtur occurrere poſſe in punctis quorum numerus erit ad numerum occurſuum rectæ cum BQG ut 2 ad 1; atque licet omnes hujus ordinis rectæ occurrere poſſint in punctis numero infinitis, aliquarum tamen interſectiones cum rectis poſſunt eſſe infinite plures interſectionibus aliarum: eæ vero quarum interſectiones ſunt plures ex aliis ſimplicioribus per *Prop.* præcedentes deduci poſſunt. Sit in deſcriptione *Prop.* 3. (*Fig.* 15.) BQC Logiſtica anguli FCO, KSH recti & curvæ motu puncti P deſcriptæ (ſi poli C, S ſint in Aſſymptoto ſitque CS $= b$, CM $= x$, PM $= y$, & ſubtangens Logiſticæ æqualis a) æquatio erit $\overline{aby - 2axy - bxy + x^2y} \times x = \overline{b - x} \times axy$. Quæ proinde conſtrui poteſt ex areis Hyperbolæ conicæ & Hyperboliſmi Hyperbolæ Lineæ tertii Ordinis ſpeciei 59.

SECTIO III.

De ſeriebus curvarum infinitis quæ ex una data tangentium ope deduci poſſunt.

DEſcriptio curvarum quam præcedentibus Sectionibus demonſtravimus eſt generaliſſima & ſi methodi univerſalitas ſimul reſpiciatur ſimpliciſſima haberi poteſt; aliæ apud Geometras paſſim reperiuntur curvarum deſcriptiones quæ ſimpliciores nonnunquam exhibent methodos Lineas eaſdem curvas deſcribendi licet minus univerſales; eæ vero vagæ plerumque ſunt nec ad unum Linearum genus deſcribendum aut ad aliquas omnium generum delineandas ſufficiunt, ſed abſque ullo ordine hic unam aliquam particularem Lineam illic aliam arripiunt, & nonnunquam faciliorem exhibent deſcriptionem Linearum altioris ordinis quam aliarum ordinis inferioris. Hæ deſcrip-

deſcriptiones licet minus regulares nequaquam tamen ſunt in Geometria inutiles nec quidem injucundæ, cum ea quæ ad intellectum oblectandum conducant maxime neque varia ſint nimium nec quidem nimium generalia aut ſimplicia eſſe poſſint. Hujuſmodi Linearum deſcriptiones variæ à variis excogitatæ ſunt, quarum aliquæ ad Propoſitiones univerſales prius demonſtratas reduci poſſunt; & quiſque methodum quæ ſibi arridet aut inſigniora theoremata ſuppeditat excolere ſtudet. Cum vero Geometria quæ curvas ad datum centrum relatas contemplatur in Philoſophia naturali ad motus corporum & vires evolvendas facilius applicari poſſit, ſicuti in præcedentibus curvarum characteres ex æquatione deſignante relationem ordinatim applicatæ ad abſciſſam axis alicujus deduximus, hac ſectione conſiderabimus curvas tanquam ad punctum quodvis datum relatas ex quo ad omnia circumferentiæ puncta radii undique educuntur & ſimul perpendicula in illorum punctorum tangentes demittuntur, & rationem radii ad perpendiculum tanquam curvæ characterem uſurpabimus. Detur curva BL*l* & punctum S in ejus plano; ex puncto S (*Fig.* 16.) concipiemus rectas normales ſemper cadere in omnes tangentes LP, & ex data curva dabitur ratio radii SL ad perpendiculum SP in terminis finitis, ſive curva ſit geometrica ſive mechanica primi generis, in quibus nimirum ratio momenti ordinatæ ad momentum abſciſſæ exprimi poteſt quantitatibus finitis. Quippe deſignant x, y abſciſſam & ordinatam curvæ cujuſvis, m & n finitas quaſvis quantitates, & ſi $x : y :: m : n$, erit SL : SP :: SL $\times \sqrt{m^2+n^2} : my-nx$, adeoque ratio perpendiculi SL ad SP exprimi poterit quantitatibus finitis, cumque ſit SP $= \frac{my-nx}{\sqrt{m^2+n^2}}$ facile erit ex curvæ æquatione ad axem quemvis data rationem radii SL ad SP invenire, & vice verſa ex ea ratione data curvæ æquatio deduci poteſt, ſed difficilius quidem. Qua vero ratione in hoc curvas conſiderandi modo complexiores ex ſimplicioribus pauciſſimarum rectarum ope conſtrui poterint ex Propoſitionibus ſequentibus patebit; in quibus demonſtrabimus ſimplices deſcriptiones Linearum aliquarum quæ methodis prius demonſtratis non niſi difficili rectarum & angulorum apparatu delineari poſſunt.

PROP.

PROP. IX.

Detur punctum S *in plano datæ curvæ* BL*l* *invenire Lineam curvam ductam per concurſus tangentium curvæ datæ cum perpendicularibus ex* S *in tangentes demiſſis.*

CUrvam datam in punctis L & *l* quamproximis contingant rectæ LP & *lp*; in quas ex dato puncto S demittantur normales SP & S*p* Tangentibus occurrentes in punctis P & *p*; ducantur SL, S*l*, P*p* ſint *lo*, *pn* perpendiculares in SL, SP, eritque L*ol* triangulum elementare curvæ BL*l* & P*np* triangulum elementare curvæ quæſitæ. Triangula vero SLP & L*ol* ſimilia ſunt ob rectos SPL, L*ol* & angulum SL*l* utriſque communem; ſimilia etiam ſunt triangula S*np* & P*n*L ob verticales S*np*, P*n*L & rectos SPL, S*p*L; proinde erit P*n* : *pn* :: L*n* : S*n*, adeoque ob angulos S*n*L, P*np* æquales, erunt triangula S*n*L, P*np* ſimilia per *Elem.* & ſi SR ſit perpendicularis in P*p* productam ob triangula æquiangula P*np*, SPR ſimilia erunt triangula P*np*, L*ol*, SPR & S*n*L; adeoque cum eadem ſit ratio P*n* ad *pn* quæ L*o* ad *lo* & eadem ſit ratio SP ad SR quæ SL ad SP manifeſtum eſt ex data ratione SL ad SP in curva BL*l* dari rationem SP ad SR in curva quæſita quæ per puncta P ducitur.

Corol. I. Similiter ex curva quam P perpetuo tangit alia nova perpendiculares in ejus tangentes demittendo conſtrui poteſt, & ex hac tertia; unde ſeries curvarum infinita ex una quavis deduci poteſt in unica poſitione puncti S, & triangula quorum latera ſunt ſemper radii curvarum, perpendicula in tangentes & partes tangentium radiis & perpendiculis interceptæ erunt ſibi mutuo ſemper ſimilia. Proinde data æquatione radiali (ſeu ratione radii ad perpendiculum) unius in ſerie dabuntur ſimiles æquationes omnium.

Corol. II. Sicut autem in *Prop.* ex curva BL*l* conſtruximus curvam EP*p* ex EP*p* contraria ratione conſtrui poſſet curva BLC; erigendo nim. perpendiculares in radios SP & S*p* ſibi mutuo occurrentes in L; vel facilius, ſi detur angulus SP*p* tangentis cum radio, ducendo SL ita ut angulus PSL ſit illius anguli SP*p* complementum ad rectum, & ad P normalem radio SP erigendo rectæ SL occurrentem in L; curva enim per

omnia

omnia hujuſmodi puncta I, ducta erit BL*l*, ex qua ſimili ratione alia conſtrui poteſt perpendicula ad L & *l* radiis SL & S*l* erigendo, & eorum occurſus notando. Unde ex quavis curva ſeries utrinque infinita curvarum conſtrui poteſt puncto S manente immoto; quod ſi locus puncti S mutetur ſimul mutabitur integra ſeries. Unde in quavis curva infinitæ ſunt poſitiones puncti S ex quibus ſingulis infinitæ curvarum ſeries deduci poſſunt.

Corol. III. Quoniam angulus SP*p* = SL*l*, poſtquam tangens ducitur uni curvæ facile eſt ducere tangentes omnibus curvis ſeriei; quippe ſi conſtituatur ad punctum curvæ P ſuper radium SP angulus æqualis ipſi SLP, ejus anguli alterum crus erit curvæ tangens in puncto P. Unde poſtquam curva quam P perpetuo tangit deſcripta eſt, facile abſque ulteriori tangentium ope reliquæ in ſerie conſtrui poterunt. Sic curva quam R concurſus perpendiculorum ex S in tangentes curvæ EP*p* demiſſorum cum tangentibus iiſdem perpetuo tangit, facile determinatur ducendo SR angulum RSP conſtituentem æqualem PSL, & ex P in SR demittendo perpendiculum PR; id enim perpendiculum occurret rectæ SR in puncto curvæ quæſitæ.

Corol. IV. Si curva BL*l* tranſeat per S curva puncto P deſcripta etiam tranſibit per S, & omnes ſeriei infinitæ explicatæ per S quoque tranſibunt; quippe cum coincidit L cum S evaneſcunt SL & SP, adeoque P coincidit cum S & L.

Corol. V. Ubi curva BL*l* evadit radio normalis curvæ omnes ſeriei in eodem puncto radio fiunt normales, & ſe mutuo contingunt. Quippe ubi BL*l* evadit normalis in radium coincidit L*l* cum *lo*, adeoque ob ſimilia triangula L*ol*, P*pn*, SLP coincidet P*p* cum *pn* & SP cum SL; quare coincidunt L, P*p*, *lo*, *pn* & fiunt omnes radio perpendiculares in eodem puncto. Si igitur una aliqua curvarum totius ſeriei habeat aſpidem, omnes in eodem puncto aſpidem habebunt & in eodem ſe mutuo tangent; quare ſi una quævis curva & tranſeat per S, & apſidem habeat omnes ex ea deductæ modo *Corol.* 1. tranſibunt per S & apſidem habebunt; nec in ullis aliis punctis præter S aut apſidem ſibi mutuo occurrere poterint.

Corol. VI. Si curva BL*l* maximam habeat finitam diſtantiam à puncto S, centro S atque ea diſtantia tanquam radio ducatur circulus, & omnes curvæ quæ ex BL methodo *Corol.* 1. deduci poſſunt licet infinitæ ſint numero circulo tamen hoc includentur; unaquæque vero extra proxime priorem jacebit, iiſque finitum intercedet intervallum: iſthuc contingit in omnibus curvis in ſe redeuntibus & quarum finita eſt longitudo.

Corol. VII. Hinc patet ſi una curvarum in ſe redeat omnes curvas ſeriei in ſe neceſſario redire; unde ſi prima ſit ovalis erunt omnes ovales & Lineæ ordinis paris. Si una abeat in infinitum crure Parabolico omnes ſimili crure abibunt in infinitum cum tangens cruris Parabolici ad

punctum maxime longinquum infinite diſtet: Unde curvæ hac methodo deductæ ex una quavis data ſunt ſibi mutuo ſimiles. Si curva BL abeat in infinitum crure Hyperbolico, curva BP erit adhuc ovalis, & ordinis paris, cum tangens cruris Hyperbolici ad punctum maxime diſtans ſit ejus Aſſymptotos. Unde manifeſtum eſt in curvarum harum ſerie unam tantum deprehendi poſſe curvam cujus crura poſſunt eſſe Hyperbolica, ſed reliquæ vel erunt ovales, vel lineæ quarum crura ſunt Parabolica.

Corol. VIII. Poſtquam ducta eſt curva quam P tangit in una poſitione puncti S reliquæ quæ ex eadem curva conſtruuntur in alia quavis poſitione puncti S ſic deſcribi poſſunt; ſit S in F & ex S exeat SPN angulus rectus cujus punctum P ducatur per curvam jam deſcriptam & ſi SN ſit perpendicularis in PN punctum N deſcribet curvam quæſitam.

Prop. X. Problema.

Sit curva BL *Circulus invenire curvam* BP.

SIT C circuli centrum *(Fig. 16.)* BCS diameter circuli quæ tranſit per datum punctum S; & curva BP deſcribi poteſt concurſu Linearum SP, LP ſi angulus datus CLP revolvatur circa punctum C & æquali motu recta SP circa polum S, ita ut SP ſemper parallela ſit cruri CL; quæque hac facili ratione geometrice conſtrui poteſt.

Centro S radio CS deſcribatur circulus HMC; ex S ad circulum educantur radii infiniti ut SM, S*m*, & ex punctis M, *m* demittantur normales MQ, *mq* in HC, ſit SF = CL, & ſumatur ſemper in radio SM Linea SP = FQ eritque P punctum in curva quæſita, & ſimiliter infinita curvæ puncta poſſunt determinari; ducatur enim LE parallela rectæ CS, eruntque triangula LEP, SQM ſimilia ob æquales angulos LEP, MSQ & rectos SQM, LPE; ſed æqualia quoque ſunt ob æquales rectas LE, CS & SM, adeoque SQ = EP & FQ = SF − SQ = CL (= SE) − EP = SP, & proinde ſi ſumatur in SM Linea SP æqualis rectæ FQ erit P in curva quæſita. Sit PR perpendicularis in SH, ſitque PR = y, SR = x & ob triangula ſimilia SPR, SMQ erit SM : SP :: SQ (= SF − FQ) : SR. Unde ſi SM = a, SF = CL = b erit $a : \sqrt{y^2 + x^2} :: b - \sqrt{y^2 + x^2} : x$, $\sqrt{y^2 + x^2}$, unde $x^4 + 2ax^3 + 2y^2 + a^2 - b^2 \times x^2$ quæ æquatio eſt ad Lineam quarti Ordinis.

Corol. I. Curva habet punctum duplex in S; ſit enim Kk ordinata circuli CMH ad abſciſſæ punctum F, & ubi M eſt vel in K vel k coincidit P cum S, adeoque rectæ SK, Sk contingunt arcus duos curvæ concurrentes in S; idem ex prima conſtructione hinc conſtat quod LP tangens circuli BL bis tranſeat per punctum S, & ex æquatione inveſtigata deduci quoque poteſt. Si S ſit intra circulum BLA curva habebit *Punctum* conjugatum in S, ſi S ſit in circulo curva erit *Cuſpidata* in S, & ſi S ſit extra circulum curva erit *Nodata* in S.

Corol. II. Sit A punctum ubi circulus BL occurrit rectæ CS, & curva erit perpendicularis rectæ CS in duobus punctis A, B. Quippe circulus BL perpendiculariter occurrit rectæ CS in punctis A, B, adeoque per *Corol.* 5. Propoſitionis præcedentis curva BP erit perpendicularis rectæ CS in iiſdem punctis. Curva BL non in infinitum excurrit ſed in ſe redit ut ex *Corol.* 7. ejuſdem Propoſitionis eſt manifeſtum.

Corol. III. Rectæ omnes per S tranſeuntes & ad curvam utrinque terminatæ ſunt ſibi mutuo & rectæ AB æquales; producatur SP donec occurrat curvæ in alio puncto G & circulo in m, ſitque mT perpendicularis in SH & ſumatur $Cf = HF$; erit $SG = FT$ & $FQ = SP$, ſed ob triangula ſimilia & æqualia SQM, STm erit $ST = SQ$ & $fT = FQ$, adeoque erit $PG = SG + SP = FT + fT = Ff = 2SF = AB$.

Corol. IV. Ob ſimilia & æqualia triangula SLE, SMQ erit $LP = MQ$; cumque $Pn = Qq$ erit trianglum $LPn = \frac{1}{2} MmqQ$, *i. e.* momentum areæ BZPLB eſt ad momentum areæ circularis CMQ ut 1 ad 2 adeoque cum ſimul incipiant fluere erit area integra $BZPLB = \frac{1}{2}CMQ$ & tota area quæ interjacet ſemicirculo ALB & curvæ BZPSA æqualis erit quadranti circuli radio CS deſcripti, & proinde tota area terminata recta BA & curva ASVB erit ſeſquupla ſemicirculi.

Corol. V. Curva rectificari poteſt in ſolo caſu quo S & A coincidunt (*Fig.* 17.) & ovalis quæ jacebat inter S & A evaneſcit & mutatur in cuſpidem. Ducantur BL, Bl, cumque parallelæ ſint CL, SP erit angulus $PSp = LCl = 2SLl$ & $lo : pn :: \frac{1}{2}SL : SP$; ſed $Pp : pn :: SL : SP$ (per *Prop.* 9.) adeoque $Pp = 2lo$, i. e. momentum curvæ BP eſt ad momentum rectæ BL ut 2 ad 1 cumque ſimul evaneſcant BP & BL in B erit ſemper $BP = 2BL$, adeoque arcus BP erit ſemper æqualis duplo chordæ arcus circuli correſpondentis.

Corol. VI. Curva hac Propoſitione deſcripta Epicyclois eſt (*Fig.* 18.) generata motu circuli (cujus radius eſt $\frac{1}{2}$CA) revolventis ſuper alium immotum ſibi æqualem centro O puncto medio inter C & S deſcriptum, Sit enim $OH = \frac{1}{2}CA$ & deſcribatur circulus HFM, ſuper quem volvatur alius circulus HDK ei æqualis, & in hujus plano ſumatur punctum R ad diſtantiam à centro æqualem $\frac{1}{2}$CS & punctum R deſcribet Lineam de qua hac Propoſitione agimus, ſi in principio deſcriptionis coincidat N cum H & R cum A. Quippe perveniat circulus revolvens ad poſitionem in *Fig.* delineatam & contingat Baſim in F, ſit-

que CF parallela rectæ CL & punctum deſcribens R erit in concurſu P tangentis LP cum perpendiculo SP, nam P*e* = OS = ER & angulus P*e*F = HOF; quippe ob tres rectas parallelas SP, OT, CL & æquales OS, OC erit TP = TL, adeoque cum rectus ſit angulus PT*e* erit triangulum *e*TP = *e*TL & *e*P = *e*L = (ob æquales & parallelas O*e*, CL) OC = CS = ER, & angulus P*e*F = F*e*L = (ob parallelas OC, *e*L) *e*OS, adeoque arcus HF = *n*F; proinde cum circulus HDN accedit ad poſitionem *n*FL coincidit EN cum *en* & R cum P; curva igitur Propoſitione hac deſcripta Epicyclois eſt generata motu puncti in plano circuli circa circulum æqualem revolventis dati: & ſi in deſcriptione Propoſitionis maneat circulus BLA manebunt circuli iidem HFM & NDK ſed erit ER ſemper æqualis $\frac{1}{2}$CS.

Corol. VII. Curvæ huic recta æqualis duci non poteſt; Quippe demonſtravit M. *Nicole* Epicycloidem hujuſmodi æqualem eſſe Longitudini Ellipſeos cujus axis major eſt CS + CA, & axis minor eſt SA in actis Academiæ *Pariſienſis* anni 1707. Ellipſis vero hæc cum axe ſua convenit cum S & A coincidunt, quo in caſu axis minor SA = *o* & longitudo curvæ æqualis evadit 2BA quemadmodum *Corol.* 5. demonſtravimus.

Corol. VIII. Aliæ vero ſunt harum curvarum proprietates non aſpernendæ; diametro CS deſcribatur circulus CGS (*Fig.* 19.) & ad curvam BP educatur Linea quævis SM occurrens circulo in G & curvæ in P, Linea PG hinc circulo illinc curva terminata erit ſemper æqualis datæ Lineæ CA. Quippe ſi ex M in CS demittatur perpendicularis MQ ſimilia erunt triangula SMQ, SCG, adeoque erit SM ad SC ut SQ ad SG, ſed SM = SC adeoque SQ = SG. Sed in demonſtratione Propoſitionis oſtendimus SP = FQ = SF + SQ = SF + SG, unde PG erit æqualis rectæ SF ſeu CA. Si igitur ex puncto in circumferentia circuli alicujus SGC poſito ad ipſum circulum educantur radii infiniti ut SM & in iis hinc & illinc ſumantur rectæ GP, G*p* æquales datæ Lineæ CA, curvæ hoc pacto delineatæ eæ erunt quas tot variis modis Propoſitione hac & *Corollariis* præcedentibus conſtruere docuimus. Ex hac curvæ proprietate patet eam eſſe conchoidem baſis circularis quam deſcripſit D. *de la Hire* in actis Academiæ *Pariſienſis* anni 1708, quam etiam tractarunt *Robervallius* & D. *Paſchal* eorum tamen nemo quantum novi eam Epicycloidem eſſe obſervavit.

Corol. IX. Hujus curvæ arcus aliqui facillime deſcribi poſſunt ducendo terminum alterum rectæ datæ G*p* (quæ ſemper tranſit per punctum S) per circulum CGS nam terminus alter *p* deſcribet arcum curvæ hujus de qua agimus; Circulus SGC biſecat omnes Lineas in S concurrentes & ad curvam utrinque terminatas.

Corol. X. Propoſitio hæc univerſaliter reddi poteſt oſtendendo *(Fig.* 20.*)* ſi angulus datus CNL angulari ſuo puncto N percurrat circulum, ſintque C, S puncta ubicunque in plano circuli poſita, & ſit SP ſemper perpendicularis in NL, Lineam quam P perpetuo tangit eſſe quarti Ordinis. Ducatur circuli ANB diameter AB per punctum C & DS ei parallela per punctum S ſit $AB = a$, $AC = e$, $AE = d$, $SE = b$, $PM = y$, $SM = x$, $NU = z$, $CU = u$, eritque $x : y :: u : z :: z + b - y : d + x - u$; adeoque erit $u = \frac{x^2 + dx + y^2 + by}{x^2 + y^2} \times x$, & proinde $x = \frac{x^2 + dx + y^2 - by}{x^2 + y^2} \times y$. Sed ex natura circuli eſt $u^2 + z^2 = ac + au - 2cu - c^2$, unde $\overline{x^2 + dx + y^2 - by}^2 = \overline{ac - c^2} \times \overline{x^2 + y^2} + \overline{a - 2c} \times \overline{x^3 + dx^2 + y^2 x - byx}$, quæ eſt æquatio ad Lineam quarti Ordinis. Curvæ quas hoc & præcedentibus *Corollariis* tractavimus eæ ſunt quæ ex circulo eadem ratione deſcribuntur qua Hyperbolam defectivam ex Recta *Lemmate* 2. *Partis* 1. deduximus; atque hæ ſimpliciſſimæ videntur eſſe Lineæ quarti Ordinis (ſi *Conchoidem* excipias) quemadmodum Hyperbolæ defectivæ pro ſimpliciſſimis haberi poterint Lineis tertii Ordinis.

PROP. XI. PROBLEMA.

Sit BL *Sectio quævis conica, invenire curvam* BP.

Caſus I. SIT BL Parabola *(Fig.* 21.*)* & Linea BP erit Hyperbola defectiva tertii Ordinis cujus punctum duplex eſt in S & Aſſymptotos unica perpendicularis eſt axi Parabolæ; curva erit *ſpeciei* 34, ſecundum enumerationem *Neutonianam*, ſi S non ſit in axe & ſit extra Parabolam, erit *ſpeciei* 35 ſi ſit in ipſa curva Parabolæ, & *ſpeciei* 36, ſi S ſit intra Parabolam. Sed ſi punctum S ſit in axe curva habebit diametrum & ſi S ſit extra Parabolam curva erit *ſpeciei* 40, ſi S ſit in vertice Parabolæ curva erit *Ciſſois Dioclea*, ſi S ſit inter verticem & focum curva erit conchoidalis quæ *Punctum* habet conjugatum ad convexitatem ſuam eritque *ſpeciei* 42; ſi denique focus ſit inter verticem figuræ & punctum S conchoidalis habebit *Punctum* conjugatum ad concavitatem ſuam eritque *ſpeciei* 43. Hæc omnia hinc facile patent quod ſi ex foco F parabolæ

rabolæ BL demittantur perpendiculares in tangentes LP, earum concurſus cum tangentibus ſint ſemper in recta BN quæ Parabolam tangit in vertice B; unde cum SP ſit perpendicularis in tangentem LP parallelæ erunt SP & FN, atque anguli FNP, SPN recti, adeoque per *Lemma* 2. *Partis* 1. Linea puncto P deſcripta erit Hyperbola defectiva tertii Ordinis habens punctum duplex in S. Per ibidem demonſtrata Hyperbola habebit diametrum ſi S ſit in axe parabolæ, ſeu in recta FA ad BN normali; & ex eodem *Lemmate* & natura Parabolæ conſtat Lineam habere *Nodum* in S quoties S exiſtit extra parabolam, *Cuſpidem* quando eſt in ipſa Parabola, & *Punctum* conjugatum cum S verſatur intra Parabolam.

Caſus II. Sit BL*l* Ellipſis, F & *f* foci, C centrum, & S punctum in plano datum ex quo perpendiculares SP ducantur in Ellipſis tangentes. Si ex foco F in tangentem LP cadant normales ut FN erit N punctum in circulo ſuper axem Ellipſeos tanquam diametrum deſcripto; nam CN = CA cum angulus FLN = NLH, & proinde FN = NH, atque ſit FC = *f*C, & proinde CN : *f*H (= A*a*) :: FC : F*f* :: 1 : 2. Unde inveſtigatio curvæ deſcriptæ per puncta P reducitur ad *Corollarium* 10. *Prop.* præcedentis ubi angulus datus FNP angulari puncto N duci ſupponitur per circuli circumferentiam & SP cadit normalis in NP; unde manifeſtum eſt Lineam deſcribi quarti Ordinis cujus æquatio in dicto *Lemmate* inveſtigatur.

Quod ſi Ellipſis evadat circulus coincidet focus F cum centro C, & curva deſcripta per puncta P erit Epicyclois quam *Prop.* præcedente fuſe tractavimus.

Caſus III. Sit BL*l* Hyperbola & ſimili plane ratiocinio oſtendi poteſt, ſi C ſit centrum, F & *f* foci Hyperbolæ ſitque recta FN perpendicularis in Tangentem EN, rectam CN eſſe æqualem dimidio axis tranſverſi Hyperbolæ, adeoque punctum N circulum tangere; unde manifeſtum eſt curvam in qua reperitur punctum P eam quoque eſſe quæ deſcribitur *Corol.* 10. *Prop.* præcedentis & proinde eſſe quarti Ordinis. Unde univerſaliter ſi curva BL*l* ſit ſectio conica ſeu ſecundi Ordinis curva BP erit quarti Ordinis, niſi quod in ſpeciali caſu quo BL eſt Parabola curva BP ſit Ordinis tertii.

Corol. Hinc ſi methodo *Corol.* 2. *Prop.* 3. ex circulo deſcribatur curva, & S ſit ubicunque in circuli plano, curva ea erit ſectio conica.

Lemma I.

Ex dato puncto S (Fig. 24.) *ad curvam* APB *duc radios quotcunque ut* SP *& ſumatur ſemper* Sπ = 2SP *punctum* π *perpetuo tanget Lineam ejuſdem ſpeciei & Ordinis ac eſt* APB; *idem dicendum eſt ſi* Sπ *ſit ad* SP *in data ratione.*

Corol.

2

Corol. Hinc ſi SR (*Fig. Prop.* 9.) cadat ſemper in LP ita ut detur angulus SRL, erit R punctum curvæ ejuſdem ſpeciei ac eſt BP.

Prop. XII.

Revolvatur Linea curva BL *ſuper ſeipſam ut baſim & punctum* S *in ejus plano datum deſcribet Lineam ejuſdem generis ac eſt ea quam* Prop. 9. *determinavimus concurſu tangentium curvæ cum perpendiculis ex* S *in tangentes demiſſis.*

Quippe ſibi *bl*L ſit una quævis poſitio curvæ revolventis & S locus puncti S in eo caſu; cum eadem Linea L*lp* utramque curvam contingat ſi ab initio deſcriptionis punctum *b* applicabatur puncto B manifeſtum eſt angulum SLP $=s$LP & sL $=$ SL adeoque & SP $=s$P & 2SP $=$ Ss; ſed per *Lemma* 2. eadem eſt ſpecies curvæ deſcriptæ motu ipſius P atque puncti cujus diſtantia a S eſt ſemper dupla diſtantiæ SP; unde conſtat curvam ejuſdem generis deſcribi motu puncti S & puncti P adeoque Epicycloidem deſcriptam motu curvæ BL ſuper ſeipſam eſſe curvam ejuſdem generis ac eſt ea *Prop.* 9. ex eadem BL conſtructa.

Corol. 1. Coincidunt igitur curvæ noſtra methodo deſcriptæ cum Epicycloidibus D. *Nicole* deſcriptis motu curvæ revolventis ſupra baſim ſibi æqualem; & hinc quoque facile conſtat hujuſmodi Epicycloidem fore Lineam Geometricam quoties curva cujus revolutione deſcribitur Geometrica eſt; cum curva BP ſit neceſſario Geometrica ſi curva BL cujus tangentium ope deſcribitur Geometrica ſit.

Corol. II. Hinc Epicycloides deſcriptæ motu ſectionis cujuſvis conicæ ſuper ſeipſam facile determinantur. Si Parabola moveatur ſuper aliam ſibi æqualem focus deſcribet Lineam rectam axi parabolæ immotæ perpendiculariter inſiſtentem ad diſtantiam à vertice æqualem diſtantiæ verticis à foco; Parabolæ vero vertex deſcribet *Diocleam Ciſſoidem*, & aliud quodvis punctum deſcribet aliquam Hyperbolarum defectivarum *Neutoni* punctum duplex habentem in ſimili puncto Parabolæ immotæ.

Corol. III. Si Ellipſis ſimili ratione ſupra Ellipſin æqualem revolvatur focus deſcribet circulum centrum habentem in altero foco cujus radius æqualis erit axi Ellipſeos. Aliud vero quodvis datum Ellipſeos plani punctum Lineam deſcribet quarti Ordinis qualem *Corol.* 10. *Prop.* 10. definivimus.

definivimus. Idem quoque de Hyperbola ſuper æqualem Hyperbolam revolvente dicendum; Focus enim alter deſcribet circulum habentem centrum in foco oppoſitæ ſectionis cujus radius erit axis tranſverſus Hyperbolæ: Aliud vero quodvis Hyperbolæ punctum deſcribet Lineam quarti Ordinis *Corol.* eodem *Prop.* 10. definitam.

Lemma II.

Dentur poſitione recta infinita SD *&* *puncta* S, A, *centro vero* S *deſcribantur circuli* ABE, DLT *& ſumatur arcus* DL *æqualis dato* AB *& in infinitis circulis eodem centro deſcriptis ſimiliter ſumantur arcus dl à recta* SD *æquales dato* AB *& curva per puncta omnia* L, *l ducta erit Spiralis Hyperbolica ſeu reciproca cujus radii erunt ſemper reciproce ut anguli quos conſtituunt cum recta data* SD.

Cum punctum D accedit ad A coincidat punctum L cum B, jungatur SB & univerſaliter radius quivis SL erit ad SB ut angulus BSA ad LSD; Arcus enim circulorum eodem centro deſcriptorum ſunt in ratione compoſita angulorum quos ſubtendunt & radiorum quibus deſcribuntur, & proinde cum ex Hypotheſi arcus AB, DL ſint æquales, radii SB, SL erunt reciproce ut anguli BSA, LSD.

Corol. I. Cum ſit do : DL :: So : SL erit DL — do $(= lo)$: Lo :: DL : SL, ſed SP : LP :: lo : Lo, unde SP : LP :: AB : SL.

Corol. II. Hinc SL $\times lo =$ AB $\times$ Lo adeoque areola 2SLl erit æqualis rectangulo datæ quantitatis AB in Dd, & proinde area radiis SB, SL & curva BL terminata erit ad rectangulum arcus dati AB in differentiam radiorum AD ut 1 ad 2.

Prop. XIII. Problema.

Ex Spirali Archimedis (Fig. 23.) *Circulo & Recta conſtruere omnes curvas in quibus trianguli* SLP *latera quævis duo ſunt ſemper ad ſe mutuo in ratione alicujus poteſtatis radii ad datam quantitatem.*

HUjus Propoſitionis ſunt tres caſus vel enim in curva conſtruenda oportet eſſe SP : LP :: SL^m : SB^m, vel LP : SL :: SL^m : SB^m vel SP : SL :: SL^m : SB^m.

Caſus I. Conſtruenda ſit imprimis curva cujus perpendiculum SP eſt ad tangentem LP in ratione poteſtatis ipſius radii SL ad datam quamvis quantitatem

quantitatem & Problema per ſpiralem *Archimedis* conſtrui poterit. Sit curva BRrS ſpiralis Archimedis eritque Ru : ru :: SB : SR. Sumatur angulus BST ad BSR ut eſt 1 ad m, & in recta ST ſumatur SL ad SB ut $SR^{\frac{1}{m}}$ ad $SB^{\frac{1}{m}}$ & L erit punctum in curva quæſita in qua erit SP : LP :: SL^m : SB^m. Quippe ſi L, l & R, r ſint puncta quamproxima, & lo, ru ſint perpendicula, ex l, r in radios SL, SR demiſſa, erit Lo : Ru :: $\frac{1}{m} SR^{\frac{1-m}{m}} : SB^{\frac{1-m}{m}}$. Cumque BSR : BSL :: m : 1 erit RSr : LSl :: m : 1 adeoque erit lo : ru :: SL : $m \times$ SR :: $SR^{\frac{1}{m}} : m \times SR \times SB^{\frac{1-m}{m}} :: \frac{1}{m} \times SR^{\frac{1-m}{m}} : SB^{\frac{1-m}{m}}$ :: Lo : Ru. Unde Lo : Ru :: lo : ru adeoque triangula Rur, Lol, SLP ſimilia erunt & SP : LP :: ru : Ru :: (ex natura Spiralis *Archimedeæ*) SR : SB :: SL^m : SB^m; proinde punctum L erit in curva quæſita cujus tangens erit ad perpendiculum ut data quantitas ad poteſtatem *(m)* radii.

Caſus II. Oporteat ſecundo conſtruere curvas quarum tangentes LP ſunt ad radios SL in ratione poteſtatis alicujus *(m)* radii SL ad datam quantitatem. Sit in *Prop.* 9. BL Spiralis Hyperbolica *Lemmatis* præcedentis quæ per caſum 1. ex Spirali *Archimedis* conſtruitur ſumendo BST = BSR & SL ad SB ut SB ad SR, eritque SL : LP :: SP : TP ſed per *Corol.* 1. *Lemmatis* præcedentis eſt SL : LP :: AB : SP, & proinde erit SP : TP (:: Pp : Pn) :: AB : SP. Unde in curva BP tangentes TP ſunt ad radios SP in ratione ipſorum radiorum ad quantitatem datam AB; & in hac curva ſi radii ſumantur in progreſſione Geometrica erunt longitudines curvarum in progreſſione Arithmetica, adeoque una eſt curvarum quas D. *Varignon* Logarithmicas appellat; ex hac curva omnes quarum radii ſunt ad tangentes ut data quantitas ad poteſtatem aliquam radii (ſive quarum tangentes LP ſunt ſemper in ratione alicujus poteſtatis radii) conſtrui poſſunt, eadem ratione qua conſtruximus in caſu 1. curvas quarum SP : LP :: SL^m : SB^m ex Spirali *Archimedis*.

Caſus III. In circulo *(Fig.* 25.*)* SRB ob rectum angulum SRB ſimilia erunt triangula SBR, Rur, adeoque erit Rr : ru :: SB : SR; unde per conſtructionem *Caſus* 1. omnes curvæ quarum SL : SP :: SB^m : SL^m ſive quarum perpendicula SP ſunt ſemper in ratione poteſtatis alicujus radii ex circulo deduci poſſunt, ſumendo ſcil. angulum BST ad BSR ut 1 ad m & SL ad SB ut $SR^{\frac{1}{m}}$ ad $SB^{\frac{1}{m}}$. Quod ſi loco circuli SRB ſumatur *(Fig.* 26.*)* recta BR (cum triangula Rur, SBR ſimilia ſint & Rr : ru :: SR : SB) eadem conſtructione deſcribi poterint curvæ omnes quarum SP : SL :: SB^m : SL^m.

P *Corol.* I.

Corol. I. Omnes curvæ quarum LP eſt ad SP vel ad SL in ratione poteſtatis alicujus radii ad datam quantitatem ſunt mechanicæ; cum ſpirales quarum ope conſtruuntur Mechanicæ ſint.

Corol. II. Curvæ omnes caſu 2. deſcriptæ ſi *m* non ſit æqualis unitati rectificari poſſunt.

Corol. III. Curvæ omnes quarum SP eſt ad SL in ratione poteſtatis alicujus radii ad datam quantitatem ſunt Geometricæ, cum circulus & recta quarum ope deſcribuntur Geometricæ ſint. Eæ quoque omnes habent apſidem ſeu punctum ubi radio fiunt normales; quippe cum $SL = SB$ evadit $SL^m = SB^m$, $SB = SP$ & $Ll = lo$ adeoque curva radio in eo puncto evadit perpendicularis; eæ quoque curvæ hoc habent ſpeciale quod ſingulæ quæ ex una aliqua conſtruuntur methodo *Prop.* 9. ſint ejuſdem generis, *i.e.* ſi in curva BL ſit $SL : SP :: SB^m : SL^m$ erit quoque in curva BP radius SP ad perpendiculum ST in ratione alicujus poteſtatis ipſius SP ad datam quantitatem; unde ex una data infinitæ ejuſdem generis conſtrui poſſunt, ut ex *Prop.* ſequenti luculentius patebit.

Prop. XIV. Problema.

Sit curva BLl *talis ut* SL *ſemper ſit ad perpendiculum* SP *in ratione quantitatis datæ ad poteſtatem aliquam* (*n*) *radii* SL (Fig. Prop. 9.) *invenire curvam* BP.

Cum ſit $SP = \frac{SL^{n+1}}{SB^n}$ erit $Ll : lo :: SL : SP :: SB^n : SL^n$, ſed $Pp : pn :: SL : SP :: SB^n : SL^n ::$ (cum $SL = SB^{\frac{n}{n+1}} SP^{\frac{n}{n+1}}$) $SB^n : SB^{\frac{nn}{n+1}} SP^{\frac{n}{n+1}} :: SB^{\frac{n}{n+1}} :: SP^{\frac{n}{n+1}}$. Unde ſi ST ſit perpendicularis in lineam tangentem curvam BP ad punctum P erit $SP : ST :: SB^{\frac{n}{n+1}} : SP^{\frac{n}{n+1}}$, adeoq; $ST = \frac{SP^{\frac{2n+1}{n+1}}}{SB^{\frac{n}{n+1}}}$.

Curva vero BP ſic facile ex BL abſque tangentium ope conſtruitur, ſumatur angulus BSP ad BSL in ratione $n + 1$ ad 1 & in SP crus alte-

rum

rum iſtius anguli demittatur normalis ex puncto L; occurſus perpendicularis cum dicto crure anguli BSP erit in curva BP: Methodo hujus inverſa & contraria ex data BP conſtrui poſſet BL.

Quippe angulus PSp : SLl :: $\frac{pn}{SP} : \frac{lo}{SL}$ :: (cum $pn : lo :: Pn : Lo$) $\frac{Pn}{SP} : \frac{Lo}{SP}$, ſed $SP = \frac{SL^{n+1}}{a^n}$, adeoque P$n$ monumentum radii SP erit ad momentum Lo ut $\overline{n+1} \times SL^n$ ad SB^n adeoq; $\frac{Pn}{SP} : \frac{Lo}{SL} :: \frac{\overline{n+1}\,Lo}{SL} : \frac{Lo}{SL} ::$ $n+1 : 1$. Proinde PSp : LSl :: $n+1 : 1$, adeoque BSP : BSL :: $n+1 : 1$. Quippe cum anguli BSP, BSL ſimul evaneſcat coincidentibus una omnibus punctis B, P, & L (quoniam B eſt apſis curvæ) erunt anguli in ratione $n+1$ ad 1 cum momenta ſint in ea ratione data.

Similiter atque conſtruximus ex BL curvam BP, ex BP conſtrui poteſt alia BT cujus radius erit ad perpendiculum in tangentem ut poteſtas $\frac{n}{2n+1}$ rectæ datæ SB ad eandem poteſtatem radii, & ſi ſeries curvarum infinita hac ratione ex BL conſtruatur, ipſa vero curva BL ſit prima in ſerie & m exprimat ordinem curvæ cujuſcunque à BL, erit radius in ea curva ad perpendiculum ut $\frac{n}{mn+1}$ poteſtas rectæ datæ SB ad eandem poteſtatem radii curvæ.

Sed ſi ex curva BL ſecundum *Corol.* 2. *Prop.* 9. curva BN conſtruatur methodo priori inverſa, & ex BN alia eadem ratione deſcribatur prodibit ſimilis ſeries curvarum infinita; & ſi m curvæ cujuſvis ordinem exprimat in ſerie, ſive ejus diſtantiam à BL erit in ea radius ad perpendiculum ut poteſtas $\frac{n}{1-mn}$ datæ rectæ SB ad eandem poteſtatem radii.

Prop. XV.

Sint BP, BL, BN *proximæ in eadem ſerie curvarum ex aliqua* Caſus 3. Prop. 13. *deſcriptarum, dico Longitudinem curvæ cujuſvis* BP *eſſe ad ſummam Longitudinis* BN *penultimæ in ſerie ejuſque tangentis* LN *ut* $n+1$ *ad* 1.

QUippe per *Prop.* 9. angulus $SPp = SLl$ eſtque $LSl : PSp :: 1 : \overline{n+1}$ per *Prop.* præcedentem. Unde erit $Ll : Pp :: SL : \overline{n+1} \times SP$ ſed $Ll : lo :: SL : SP$ & $Ll : \overline{n+1} \times lo :: 1SL : \overline{n+1} \times SP$, unde $Ll : Pp :: Ll : \overline{n+1} \times lo$, & proinde $Pp = \overline{n+1} \times lo$. Sed $lo = ln - on =$ (cum $on = Ln$ ob angulos Lon & nLo rectos & æquales) $ln - Ln = ln - LN + Nn$; ſed $ln - LN$ eſt ut momentum rectæ LN, & Nn eſt ut momentum curvæ BN, unde lo eſt æqualis ſummæ momentorum curvæ BN, & tangentis LN; ſed $Pp = \overline{n+1}\, lo$, adeoque momentum curvæ BP eſt ad eandem ſummam ut $n+1$ ad 1. Cum vero BP, BN & LN omnes una evaneſcant in apſide B ubi coïncidunt puncta N, L P & B manifeſtum eſt quantitates fore in eadem ratione ſeu $BP : BN + LN :: n+1 : 1$; & $BP = \overline{n+1} \times \overline{BN+LN}$.

Corol. I. Similiter oſtendi poteſt $BT = \frac{2n+1}{n+1} \times \overline{BL+LP}$ & univerſaliter ſi dentur longitudines duarum curvarum proximarum in ſerie ex his dabuntur longitudines omnium; quippe menſura cujuſvis curvæ pendet ſemper a menſura penultimæ in ſerie; ita ut ſi BN rectificari poſſit etiam rectificari poſſit BP; atque unum curvarum par omnibus menſurandis ſufficiat.

Corol. II. Unde ſi unius curvæ BN rectificatio ſit poſſibilis, erit dimidia pars ſeriei rectis commenſurabilis; at ſi curvæ BL rectificatio ſit impoſſibilis, dimidia ſeriei pars erit rectis incommenſurabilis; ita ut ſi 0, 1, 2, 3, 4, &c. denotent numeros curvarum incipiendo à BN, eæ curvæ quarum numeri correſpondentes ſunt pares perfectæ rectificationis ſint capaces, quarum vero numeri ſunt impares rectificationem omnem reſpuant. Quod ſi neque BN neque BL ſint rectis commenſurabiles, nulla curva in ſerie integra rectis commenſurari poterit. Quod de

de BN & BL diximus, intellige de duabus quibuſvis curvis in ſerie ſibi mutuo proximis.

Corol. III. Sit S longitudo curvæ cujuſvis, Σ longitudo curvæ hanc proxime præter unam antecedentis, licet neque S nec Σ nec $S - \Sigma$ exhiberi poſſint in Lineis rectis, differentia tamen ipſius S & $\overline{n+1} \times \Sigma$, quantitas nimirum $S - \overline{n+1}\, \Sigma$, eſt ſemper in data ratione ad tangentem curvæ Σ.

Corol. IV. Arcus curvarum ſeriei rectis incommenſurabilis ſunt etiam ſibi mutuo incommenſurabiles; quippe cum eorum differentia $S - \overline{n+1}\, \Sigma$ ſit æqualis rectæ; ſi S & Σ ſibi mutuo commenſurarentur, tunc differentiæ etiam ſuæ $S - \overline{n+1}\, \Sigma$, adeoque datæ rectæ commenſurabiles eſſent.

Corol. V. Si curva tranſeat per S, tum recta LN evaneſcente in S coincidentibus punctis L, P, N & S, longitudo curvæ $S = \overline{n+1} \times \Sigma$; unde curva quævis erit ad penultimam ut $\overline{n+1}$ ad 1; ita ut totales curvarum longitudines ad primam ſeu ſimpliciſſimam aſſignabilem habeant rationem.

Prop. XVI.

Curvas ſimpliciores enumerare quarum perpendicula SP *ſunt ſemper in ratione poteſtatis alicujus radiorum* SL.

CUrvarum quarum $SL : SP :: SB^n : SL^n$ inſigniores ſunt 1°. Circulus puncto S exiſtente in ejus circumferentia *(Fig. 27.)* Quippe ob angulum S*l*B in ſemicirculo rectum, erit *l*B perpendicularis in SL & ſimilia triangula S*l*B & L*ol*, adeoque SL : SP :: SB : SL & $n = 1$.

II. Parabola exiſtente S in foco *(Fig. 28.)* quippe ex *Lemmate* 14. *Lib.* 1. *Princip.* perpendiculum ex foco S demiſſum in tangentem LP medium eſt proportionale inter SB & SL, adeoque $SL : SP :: \sqrt{SL} : \sqrt{SB}$, unde $n = -\frac{1}{2}$.

III. Huc pertinet recta Linea ſi S extra rectam ponatur, nam SB = SP unde SL : SP :: SL : SB & $n = -1$.

IV. Huc pertinet Epicyclois quæ deſcribitur revolutione circuli ſuper baſim ſibi æqualem puncto deſcribente exiſtente in circumferentia circuli, ſi datum punctum S ſupponatur eſſe ubi punctum deſcribens tangit baſim,

ſim, ut ex *Corol.* 5. *Prop.* 10. facile conſtat cum triangula (*Fig.* 17.) SPT, P*pn*, SLP, SBL ſimilia ſint, ſitque SB : SL :: SL : SP.

V. Harum quoque alia eſt Hyperbola æquilatera ſi S ſumatur in figuræ centro; quippe ſit BL recta poſitione data, S punctum extra rectam, ſitque SB perpendiculum ex S in rectam demiſſum; ex puncto dato S exeant ſemper radii SL, *sl*, & biſecet SN angulum BSL, & in recta SN ſumatur SP media proportionalis inter radium SL & datam rectam SB; eritque P in Hyperbola æquilatera cujus centrum S & vertex B.

In SN ex L cadat LN normalis; & ſimilia erunt triangula SLN, SNT, SNO, S*u*B, SPM, & quoniam SB : SP :: SP : SL, erit SB^2 : SB :: SP^2 : SL; ſed SO : SN :: SN : SL & SO : SL :: SO^2 : SN^2 :: SM^2 : SP^2, unde SB^2 : SB :: SM^2 : SO ſeu $SO = \frac{SM^2}{SB}$. Sed $SO = SB + \frac{BT}{2}$ (cum BO : BT :: LN : LT) $= \frac{SB + SL}{2} =$ (quoniam $SL = \frac{SP^2}{SB}$) $\frac{SB^2 + SP^2}{2SB}$ & proinde $\frac{SM^2}{SB} = \frac{SB^2 + SP^2}{2SB} = \frac{SB^2 + SM^2 + PM^2}{2SB}$, atque adeo $SM^2 = SB^2 + PM^2$. Sit igitur $SB = a$, $PM = y$, $SM = x$, eritque $x^2 = a^2 + y^2$; unde patet curvam eſſe Hyperbolam æquilateram cujus centrum eſt S & vertex B.

Ex hac conſtructione patet in Hyperbola eſſe perpendiculum in tangentem ex centro demiſſum ad radium ut SB^2 ad SP^2. Cum *Corol.* 5. *Prop.* 4. demonſtratum ſit ex recta BL conſtrui poſſe omnes quarum perpendicula ſunt ad radios ut SB^m ad SP^m ſumendo BSN : BSL :: 1 : *m* & $SP = a^{\frac{m-1}{m}} SL^{\frac{1}{m}}$; & in conſtructione hac BSN : BSL :: 1 : 2, $SP = a^{\frac{1}{2}} SL^{\frac{1}{2}}$ adeoque $m = 2$.

Atque hæ inſigniores ſunt curvæ quarum perpendicula ſunt in ratione poteſtatis radiorum; reliquæ complexiores ſunt & ex his deduci poſſunt. Ex harum ſingulis infinitæ curvarum ſeries conſtrui poſſunt, quarum inſigniores aliquæ ſpecialiorem conſiderationem merentur; initium vero fiat à circulo.

PROP.

Prop. XVII.

Sit curva BL *circulus*, S *punctum in ejus circumferentia datum, invenire curvam* BP *&* *ſeriem univerſam quæ ex* BL *ſecundum* Prop. 9. *conſtrui poteſt.*

I. Ex Articulo 1. *Prop.* præcedentis conſtat in circulo eſſe SL : SP :: SB : SL & $n = 1$. Sed per *Prop.* 14. SP : ST :: $SB^{\frac{n}{n+1}} : SP^{\frac{n}{n+1}}$ :: $SB^{\frac{1}{1+1}} : SP^{\frac{1}{1+1}}$ :: $SB^{\frac{1}{2}} : SP^{\frac{1}{2}}$, unde ſi r & p denotent univerſaliter radium lineæ curvæ & perpendiculum in tangentem ex puncto S demiſſum, ſitque SB $= a$, erit in curva BP $r : p : a^{\frac{1}{2}} : r^{\frac{1}{2}}$. Hæc vero curva Epicyclois eſt deſcripta motu circuli ſuper baſim æqualem ubi punctum deſcribens ponitur in circumferentia circuli ut dupliciter conſtat tum ex *Prop.* 10. *Corol.* 5. tum ex *Prop.* 12. Curva ex qua circulus formatur, ut BP ex circulo, eſt ipſum punctum B; nam ſi in lineas omnes tangentes punctum B (*i.e.* ex eo puncto exeuntes) normales demittantur ex S concurſus omnium tangentium & perpendicularium erunt in circulo ut ex *Elementis* conſtat, coincidunt igitur B & N & proinde BP $= \overline{n+1} \times \overline{BN + LN}$ $= 2LN = 2BL$, unde arcus Epicycloidis BP erit æqualis duplo chordæ correſpondentis in circulo BL, adeoque integra curva BPS erit dupla diametri SB.

II. Eadem ratione qua conſtruximus Epicycloidem ex circulo deſcribatur SπB ex Epicycloide SPB, conſtituendo nim. angulum SPπ$=$SLP, & ex S in lineam Pπ Epicycloidis tangentem demittendo perpendicularem Sπ; ex *Prop.* 14. ſi curva BL ſit ejuſmodi ut $r : p :: a^n : r^n$ manifeſtum eſt in curva Sπ ſecunda à circulo fore $r : p : a^{\frac{n}{2n+1}} : r^{\frac{n}{2n+1}}$ :: (cum $n = 1$) $a^{\frac{1}{3}} : r^{\frac{1}{3}}$. Unde Sπ : SR :: $SB^{\frac{1}{3}} : S\pi^{\frac{1}{3}}$, adeoque SR $= \frac{S\pi^{\frac{4}{3}}}{SB^{\frac{1}{3}}}$.

Ex

Ex L in SB cadat LG normalis & ex G demittatur GM perpendicularis in SL; & ſi ſumatur angulus BSП = 3BSL & SП = SM erit П in eadem curva quam conſtruximus ex Epicycloide.

Sit *m* index æquationis curvæ BP, & ex *Prop.* 6. conſtabit BП = $\overline{m+1} \times \overline{BL+LP}$ = (cum $m = \frac{1}{2}$) $\frac{3}{2}\,\overline{BL+LP} = \frac{3}{2}\,\overline{BL+LG}$; unde BП eſt ſeſquupla ſummæ arcus correſpondentis ejuſque ſinus recti, ſeu ſeſquupla ordinatæ Cycloidis (cujus circulus genetrix eſt BLS & vertex B) ad punctum G. Quod ſi centro S radio SD = $\frac{3}{2}$ CS deſcribatur circulus occurrens rectæ SP in H, & HQ demittatur perpendicularis in SB, erit BП = DH + HQ; nam ob æquales angulos BSP, BCL ſimilia ſunt ſegmenta BCLB & DSHD, adeoque DH : BL :: DS : CS :: 3 : 2, & CL : SD :: LG : HQ, adeoque HQ = $\frac{3}{2}$ LG, unde BL = $\frac{3}{2}\,\overline{BL+LG}$ = DH + HQ. Hinc 1. Longitudo arcus BП eſt ſumma arcus circuli DH ejuſque ſinus recti HQ, & proinde curvæ BПS arcus non poſſunt rectis commenſurari, ſed earum menſuræ Rectificationem circuli poſtulant; 2. Neque arcubus poſſunt circuli commenſurari, nam ſi BП commenſurabiles eſſent arcubus circuli BLS poſſent etiam rectis commenſurari, cum BП = $\frac{3}{2}$ BL + $\frac{3}{2}$ LG. 3. Licet arcus BП & arcus circuli DH rectis ſint incommenſurabiles, eorum tamen differentia eſt æqualis aſſignabili rectæ HQ. 4. Quando coincidunt L, P, П & S evaneſcit LG & BП = $\frac{3}{2}\,\overline{BL+LG} = \frac{3}{2}$ BLS, adeoque tota curva BLS ſeſquupla erit ſemicirculi BLS, & proinde licet arcus quivis curvæ BПS ſint circuli arcubus incommenſurabiles, eſt tamen integra curva ad circulum in ratione 3 ad 2. 5. Nulla pars curvæ aſſignabilis eſt commenſurabilis totæ nec curva in data ratione ſecari poteſt; nam ſi BПS in aſſignabili quavis ratione ſecari poſſet conſtaret quadratura circuli. Quippe ſi BП = $\frac{BПS}{m}$ & BL = $\frac{BLS}{n}$, eſſet $\frac{BПS}{m} = \frac{3}{2}\,\overline{BL+LG} = \frac{3}{2n}$ BLS + $\frac{3}{2}$ LG, ſed $\frac{BПS}{m} = \frac{3}{2m}$ BLS, adeoque eſſet $\frac{3}{2m}$ BLS = $\frac{3}{2n}$ BLS + $\frac{3}{2}$ LG, & $\frac{3\times\overline{n-m}}{2mn}$ BLS = $\frac{3}{2}$ LG, adeoque BLS = $\frac{mn\,LG}{n-m}$, & proinde ſemicirculus BLS eſſet ad rectam LG in data ratione *mn* ad *n—m*.

Habemus

Habemus igitur curvam Geometricam cujus biſectione aut triſectione prodiret recta circulo æqualis. Sic ſi *e. g.* ſit Π punctum curvæ BΠS medium ita ut BΠ $=$ SΠ, & arcus circuli correſpondens ſit BL, cujus duplo æqualis ſit arcus Bκ, erit arcus circuli Sκ $=$ 2LG. Si Π ſit punctum curvæ BΠS tale ut BΠ $= \frac{B\Pi S}{3}$, erit arcus circuli SΔ $= \frac{3}{2}$LG; & generatim ſi BΠ $= \frac{B\Pi S}{m}$ & BU $= m$BL erit arcus circuli SU $= m$LG.

III. Ex curva BΠS dein conſtrui ſupponatur BR ad modum *Prop.* 1. eritq; $r : p :: a^{\frac{n}{3n+1}} : r^{\frac{n}{3n+1}} :: a^{\frac{1}{4}} : r^{\frac{1}{4}}$, ut ex æquatione generali prius *Prop.* 14. demonſtrata liquet. Cum vero index æquationis curvæ BΠS ſit $\frac{1}{3}$ patet ex *Prop.* 6. BR $= \frac{4}{3}\overline{BP + P\Pi} = \frac{4}{3}\overline{2BL + P\Pi}$. Unde manifeſtum eſt curvam hanc rectificari poſſe & eſſe ad 2BL $+$ GM ut 4 ad 3. Unde tota curvæ longitudo BRS $= \frac{8}{3}$ SB.

Ex M in SB cadat normalis MK, & ſi angulus BSR $=$ 4BSL, SR $=$ SK erit R in curva deſcripta ut ſatis eſt manifeſtum; cum triangula BSL, LSP, PSΠ, ΠSR, ſint reſpective æqualia triangulis BSL, LSG, GSM, MSK. Proinde ſi ſuper SB deſcribatur curva quarti Ordinis per puncta M tranſiens cujus æquatio erit $ax^2 = \overline{y^2 + x^2}^2$ ſive $y^2 + x^2 = \sqrt{ax^2}$ ubi $y =$ KM, SM $= x$ & SB $= a$; erit area curva BΠ & rectis SΠ, SB intercepta BSΠB $=$ 3SMBS, cum angulus BSΠ $=$ 3BSM & ΠSπ $=$ 3MSm ac SΠ $=$ SM; adeoque tota area BΠzSB $=$ 3BMoS. Area vero incluſa curvis duabus BΠ & BR ac recta ΠR erit æqualis dimidio areæ BMK, nam ΠR $=$ MK, & ΠRr $= \frac{MKkm}{2}$. Ergo BSRB $= \frac{BMK}{2} + 3BSMB + SKM = 4SKM + \frac{7}{2}$ BMK, adeoque area tota incluſa curva integra BRS & diametro SB erit æqualis $\frac{7}{2}$ BMOS. Sed ex D. *Neutoni* quadratura curvarum *Formula* 4. *Caſus* 7. area BMS inveſtigari poteſt, atque inde areæ BΠS, BRS innoteſcent.

IV. Eadem ratione ſeries curvarum infinita continuari poteſt; & ſi n exprimat diſtantiam curvæ cujuſvis ſeriei à circulo erit in ea curva $r : p :: a^{\frac{1}{n}} : r^{\frac{1}{n}}$ ſive $p = r^{\frac{n+1}{n}} a^{-\frac{1}{n}}$ de qua ſerie obſervare licet 1. indicem radii eſſe numerum fractum cujus Numerator eſt unitas Denominator numerus

dentis patebit eandem eſſe continuatam per indices negativos. In hac ſerie primæ ſunt recta atque Parabola; & proinde ſimiliter ac in priori erit dimidia pars hujus ſeriei rectis commenſurabilis & dimidia à rectificatione Parabolæ ſeu Quadratura Hyperbolæ pendebit; omnes ſcil. quarum indices denominatorem habent numerum imparem rectis commenſurari poſſunt; quæ vero denominatorem habent numerum parem rectis & ſibi mutuo ſunt incommenſurabiles.

Prop. XIX.

Sit BL *Hyperbola æquilatera ſitque ejus centrum in* S *requiruntur curvæ omnes quæ ex ea conſtrui poſſunt eadem methodo.*

I. DEmonſtravimus in *Articulo* 5. *Prop.* 7. (*Fig.* 32.) æquationem radialem Hyperbolæ ad centrum fore (ſi a ſit ſemiaxis tranſverſus) $r : p :: r^2 : a^2$: adeoque $n = -2$. Unde æquatio curvæ BP erit $r : p :: a^{\frac{n}{n+1}} : a^{\frac{n}{n+1}} :: a^{\frac{-2}{-1}} : r^{\frac{-2}{-1}} :: a^2 : r^2$. Quæ curva eadem plane ratione ex circulo deſcribi poteſt qua dicto *Articulo Prop.* 16. Hyperbolam ex data recta conſtruximus.

Sit SLB circulus & S punctum in circumferentia datum ex quo ad circulum ducatur radius SL; biſecetur angulus BSL recta SN atque ſumatur in SN recta SP media proportionalis inter SB & SL; eritque P punctum curvæ in qua erit $r : p :: a^2 : r^2$ quæque ex Hyperbola conſtruitur per *Prop.* 9. Hæc curva eſt quarti Ordinis cumque ab ea pendeat ſeriei hujus dimidia pars erit operæ pretium ejus æquationem ad axem SB indagare; ſit LQT normalis ex L in SN demiſſa, occurrens diametro SB in T; ſint quoque LF, QN, & PM perpendiculares in SB. His poſitis erit SL : SP :: SP : SB, & proinde SQ : SM :: SP : SB, ſed SQ : SP :: SN : SM. Et proinde SN : SM :: SM : SB, ſed $SN = SF + \frac{ST - SF}{2}$ & SF : SL :: SL : SB, unde prodibit $SP^4 + SB^2 \times SP^2 = 2SB^2 \times SM^2$ ſed $SP^2 = SM^2 + PM^2$ adeoque ſi $PM = y$ $SM = x$ erit $\overline{x^2 + y^2}^2 + a^2x^2 + a^2y^2 = 2a^2x^2$, adeoque $\overline{x^2 + y^2}^2 = a^2x^2 - a^2y^2$, patet igitur curvam eſſe quarti Ordinis, & punctum duplex habere in S.

2. Ex hac curva alia deſcribi poteſt in qua erit $r : p :: a^{\frac{2}{3}} : r^{\frac{2}{3}}$ & æquationes curvarum quæ methodo directa ex Hyperbola deduci poſſunt ſeriem conſtituunt, ubi indices ſemper habent numerum binarium numeratorem & numeros impares 1, 3, 5, 7, &c. denominatores. Omnes curvæ quarum indicum denominatores ſunt in ſerie — 1, 3, 7, 11, &c. exhiberi poſſunt in rectis & arcubus Hyperbolicis; quæ vero denominatores habent 1, 5, 9, 13, &c. exhiberi poſſunt in arcubus curvæ modo explicatæ & in rectis, ut ex *Prop.* 15. ſatis eſt manifeſtum.

3. Ex Hyperbola methodo inverſa ea nim. *Corol.* 2. *Prop.* 9. deducitur ſimilis ſeries curvarum ubi indices æquationum erunt in eodem ordine ac in priori ſed negativæ. Harum omnes quarum indicum denominatores ſunt in ſerie 1, 5, 7, &c. exhiberi poſſunt in arcubus Hyperbolicis & rectis; reliquarum menſura pendet à curva prius explicata.

Schol. Series prioribus propoſitionibus inveſtigatæ includunt omnes æquationes quarum indices poſſunt habere numeratorem unitatem, & ex hac *Prop.* demonſtratæ omnes includunt quæ poſſunt habere numeratorem numerum binarium, quæ ad priores reduci non poſſunt. Proinde jam exhibuimus omnes curvas ſimpliciores quarum æquationes ad generalem $r : p :: a^n : r^n$ reduci poſſunt. At ſi aliæ quærantur curvæ quæ alias exhibere poſſint ſimiles ſeries, innumeræ ex *Caſu* 3. *Prop.* 13. peti poſſunt.

E.g. Si quæratur ſeries curvarum quarum indices numeratorem habere debeant numerum ternarium. Sit BL vel recta vel circulus, ſitque angulus $BSN = \frac{BSL}{3}$ & in SN ſumatur SP *(Fig.* 25, 26.*)* primum duorum mediorum proportionalium inter SB & SL; eritque P in curva cujus æquatio ad S erit $r : p :: r^3 : a^3$ ſi BL ſit recta; ſed $r : p :: a^3 : r^3$ ſi BL ſit circulus, & S in circumferentia; ex his duabus curvis quatuor ſeries deduci poſſunt in quibus indicum denominatores ſequentur ordinem numerorum Arithmeticæ proportionalium quorum differentia erit numerus ternarius. Omnium vero harum curvarum arcus exhiberi poſſunt in rectis & arcubus curvarum duarum ope rectæ & circuli deſcriptarum: Atque his conſtruximus curvas omnes quarum indicum numeratores poſſunt eſſe æquales numero ternario denominatores vero numeri quicunque integri.

PROP.

Prop. XX.

Circa centrum S (Fig. 33.) *deſcribatur radio quovis* SB *circulus* BQA *occurrens rectæ* SP *productæ in* Q, *ad omnia puncta* Q *erigantur perpendicula ad planum figuræ æqualia ipſi* SP, *&* *ſuperficies Cylindrica hac ratione conſtructa æqualis erit rectangulo radii in ſummam curvæ* BL *ejuſqae tangentis* LP.

QUippe cum $pn = lp - ln = lp - LP + LP - ln = L\dot{P} + L\dot{l}$ cumque (ſi radius SB $= a$ BQ $= c$ & Q$q = \dot{c}$) ſit $pn : \dot{c} :: SP : a$ erit $pn = \frac{SP \times \dot{c}}{a} = L\dot{P} + L\dot{l}$; ſed SP$\times \dot{c}$ eſt momentum ſpatii cylindrici conſtructi erigendo SP perpendicularem plano figuræ ad Q; eſt quoque $L\dot{P}$ momentum rectæ LP & $L\dot{l}$ momentum curvæ; unde cum ſimul evaneſcant ſpatium cylindricum recta LP & curva BL in B, ſintque curvæ BL & rectæ LP momenta ducta in a æqualia momento ſpatii cylindrici, erit ſpatium cylindricum æquale rectangulo radii in ſummam vel differentiam curvæ BL ejuſque tangentis LP.

Corol. I. Hinc ſi curva BL poſſit rectificari ſpatium illud cylindricum quadrari poterit; unde plures deduci poſſunt conſtructiones ſpatiorum cylindricorum ſimilium iis quas exhibet D. *de la Hire* in actis Academiæ *Pariſienſis* Anni 1708; atque his analogæ Spatiorum cylindricorum quadraturæ plures olim ab eximio elapſi ævi Geometra D. *Jacobo Gregorio Abredonenſi* pulcherrime demonſtratæ ſunt.

Corol. II. Si SPB ſit circulus erit ſpatium cylindricum ſecundum Propoſitionem ſuper ſemicirculum radio SB deſcriptum æquale duplo quadrati rectæ SB; & ſpatii portio quævis ſuper BQ conſtructa æqualis erit rectangulo diametri SB in chordam arcus BP.

Prop.

Prop. XXI.

Sit ut in Prop. 14. *perpendiculum* SP *ſemper ut poteſtas aliqua* $(n+1)$ *radii, eritque Radius curvaturæ Lineæ* BL *in puncto* L (Fig. 34.) *ut poteſtas* $1-n$ *ejuſdem radii* SL.

SInt RL, Rl duo radii curvaturæ arcus Ll concurrentes in R, cumque SP, RL & Sp, Rl ſint perpendiculares in eaſdem rectas LP, lp adeoque ſibi mutuo parallelæ, erunt triangula LRl, pSn ſimilia & RL : Ll : : SP : pn : : SL : Pp. Sed Ll : Pp : : Lo : Pn, adeoque RL : SL : : Lo : Pn. Cum vero $SP = \frac{SL^{n+1}}{a^n}$ erit Lo : Pn : : a^n : $\overline{n+1} \times SL^n$ unde erit RL : SL : : a^n : $\overline{n+1} \times SL^n$ & proinde $RL = \frac{a^n SL^{1-n}}{n+1}$ adeoque in curvis omnibus quas tractavimus *Prop.* 14, 15, 16, 17 & 18 & 19 radii Oſcillatorii erunt ſemper ut poteſtas $1-n$ radiorum SL.

Corol. I. Hinc ſi curva ſit Epicyclois *(vid. Fig. Prop.* 17.*)* quæ ex circulo methodo *Prop.* 17. deducitur erit radius curvaturæ æqualis $\frac{2}{3}\sqrt{a \times SP}$ ſive $\frac{2}{3}SL$. Unde curva cujus evolutione deſcribitur facile conſtrui poteſt erigendo perpendicula tangenti Pπ ad puncta quævis P, & ſumendo in perpendiculo $PX = \frac{2}{3}$ SL; erit enim X in ea curva: Atque hinc oſtendi poſſet Lineam quam tanget X fore Epicycloidem ſimilem ad diametrum triplo minorem contrarie ſitam; ſed hæc ab aliis demonſtrata ſunt.

Corol. II. Quæcunque ſit curva BL ſi SA ſit radius qui perpendicularis eſt curvæ, & in ea ſemper ſumatur SM = SP, ad punctum M ſit MQ perpendicularis ſemper æqualis radio curvaturæ in puncto L, ſitque AD æqualis radio curvaturæ in puncto A erit area ADQM æqualis $\frac{1}{2}\overline{SL^2 - SA^2}$ ſi SL major ſit SA vel $\frac{1}{2}\overline{SA^2 - SL^2}$ ſi SA major ſit quam SL: Nam ſi MQ, mq ſint ordinatæ quam proximæ erit MmqQ æquale SL × Lo quoniam RL : SL : : Lo : Pn, Pn = Mm &c.

Mm & RL = MQ. Si curva tranſeat per S & ſit SE radius curvaturæ in puncto S erit area DSMQ $= \frac{1}{2}SL^2$; atque hinc infinitæ curvarum areæ perfectæ rectificationis capaces deduci poſſunt.

Sectio IV.

De Viribus quibus Corpora curvas deſcribunt circa data centra.

Prop. XXII.

Deſcribat corpus in medio non reſiſtente curvam BL Prop. 14. *in qua* SP *eſt ut poteſtas* $(n+1)$ *radii* SL *& vis centripeta qua corpus urgetur verſus punctum* S *erit ſemper reciproce ut poteſtas* $2n+3$ *diſtantiæ.*

PRoducatur in *Fig. Prop.* 14. S*l* donec occurrat tangenti LP in *m* & triangula SP*p*, L*lm* erunt ſimilia ob angulos L*lm* (= S*lp*) SP*p* & angulos PS*p*, *l*L*m* æquales, adeoque erit SP : P*p* :: L*l* : *lm*, unde $lm = \frac{Pp \times Ll}{SP}$. Sed per Theorema *Neutoni* vis centripeta eſt ſemper reciproce ut ſolidum $\frac{SL^2 \times lo^2}{lm}$ adeoque reciproce ut $SL^2 \times \frac{SP \times lo^2}{Ll \times Pp} =$ (quoniam L*l* : *lo* :: SL : SP) $SL \times SP^2 \times \frac{lo}{Pp}$. Sed demonſtravimus *Prop.* 15. eſſe $Pp = \overline{n+1} \times lo$, & proinde vis centripeta erit reciproce ut $\frac{SL \times SP^2}{n+1}$ ſive (cum ex Hypotheſi ſit $SP = \frac{SL^{n+1}}{a^n}$) ut SL^{2n+3}. In omnibus igitur curvis quarum perpendiculum SP eſt ut radii poteſtas aliqua $(n+1)$ vis centripeta qua corpus urgetur ad S erit reciproce ut poteſtas $2n+3$ diſtantiæ.

Corol. I.

Corol. I. Velocitas corporis eſt ſemper reciproce ut poteſtas $(n+1)$ diſtantiæ; nam velocitas eſt ut arcus dato tempore deſcriptus, tempus vero ut area SL*l* ſive rectangulum $SP \times Ll$, adeoque velocitas erit reciproce ut perpendiculum SP & proinde ut poteſtas $n+1$ diſtantiæ, cum $SP = \frac{SL^{n+1}}{a^n}$. Velocitas corporis circulum deſcribentis ad diſtantiam SL eſt ſemper in ſubduplicata ratione rectanguli vis centripetæ in diſtantiam per *Prop.* 4. *Lib.* 1. *Princip.* adeoque erit æqualis $\frac{a^n \times \sqrt{n+1}}{SL^{n+1}}$ ſed velocitas corporis curvam BL deſcribentis ad eandem diſtantiam SL eſt $\frac{a^n}{SL^{n+1}}$; & proinde velocitas corporis curvam BL deſcribentis eſt ad velocitatem corporis circulum ad eandem diſtantiam deſcribentis ut 1 ad $\sqrt{n+1}$ adeoque in data ratione: Atque ea eſt quam corpus cadendo ex infinita diſtantia eadem vi centripeta ad altitudinem SL acquireret.

Corol. II. Hinc ſi (*Fig.* 27.) BL ſit circulus & centrum virium ſit in circumferentia, erit $SP = \frac{SL^2}{SB}$ adeoque $n = 1$ & vis centripeta reciproce ut Quadrato-cubus diſtantiæ, cum $2n+3 = 5$; velocitas erit reciproce ut quadratum diſtantiæ, eritque ad velocitatem corporis circulum ad eandem diſtantiam deſcribentis ut 1 ad $\sqrt{2}$.

Corol. III. Sit BL Epicyclois quam fuſe tractavimus *Prop.* 10. cujus punctum deſcribens eſt in circuli circumferentia; & ſi centrum virium ſit ubi punctum deſcribens tangit baſim erit SP (conferatur *Articulus* 1. *Prop.* 17.) $= \frac{SL^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ adeoque $n = \frac{1}{2}$ & $2n+3 = 4$; proinde vis centripeta corporis Epicycloidem dictam deſcribentis erit reciproce ut Quadrato-quadratum diſtantiæ, velocitas in ſeſquiplicata ratione diſtantiæ & ubique ad velocitatem corporis circulum ad eandem diſtantiam deſcribentis ut $\sqrt{2}$ ad $\sqrt{3}$. Si BL alia aliqua ſit curvarum ſeriei infinitæ *Prop.* 17. ex circulo deductæ erit vis centripeta inter rationem reciprocam triplicatam & quadruplicatam & ſemper major triplicata.

Corol. IV. Sit BL recta (*Fig.* 31.) & $n = -1$ & vis centripeta $\frac{a^{2n} \times \overline{n+1}}{SL^{2n+1}} = \frac{a^{2n} \times 1 - 1}{SL^{2n+3}} = 0$. Sit BL Parabola & $SP = a^{\frac{1}{2}} \times SL^{\frac{1}{2}}$, unde $n = -\frac{1}{2}$ & $2n+3 = 2$, adeoque vis centripeta erit reciproce aut quadratum diſtantiæ, velocitas in ſubduplicata ratione diſtantiæ & ad velocitatem corporis circulum ad eandem diſtantiam deſcribentis ut $\sqrt{2}$ ad 1. Si BL ſit

R alia

alia aliqua curvarum ex recta *Prop.* 18. conſtructarum vis centripeta erit ſemper inter rationem duplicatam & triplicatam & ſemper minor triplicata cui continuo appropinquat.

Corol. V. Sit BL Spiralis quæ ſemper radiis occurrit in eodem angulo eritque $SP = \frac{a \times SL}{b}$, (*Fig.* 35.) $n = 0$, & $2n + 3 = 3$; proinde corporis Spiralem hanc deſcribentis vis centripeta eſt reciproce ut Cubus diſtantiæ. Et univerſaliter ſi detur aliqua poteſtas radii (*m*) quæratur vero curva aliqua in qua vis centripeta ei poteſtati erit proportionalis, ſit $n = \frac{m-3}{2}$ & ſi curva conſtruatur ex *Caſu* 3. *Prop.* 14. in qua $SP : SL :: SL^n : a^n$ vis centripeta in ea curva erit reciproce ut poteſtas (*m*) diſtantiæ.

Corol. VI. Hinc quoque ſi corpus deſcribat Hyperbolam æquilateram vis centrifuga qua urgetur à centro erit ut diſtantia; nam in hoc caſu per *Prop.* 16. *Articulum* 5. $SL : SP :: SL^2 : SB^2$ adeoque $n = -2$, & $2n + 3 = -1$, & proinde vis centripeta erit ut diſtantia, qui caſus eſt particularis *Prop.* 10. *Lib.* 1. *Princip.*

Schol. Cum vis centripeta ſit ſemper reciproce ut ſolidum $SL \times SP^2 \times \frac{Lo}{Pp}$, hinc in alia quavis curva lex vis centripetæ determinari poteſt inveſtigando curvam BP quæ *Prop.* 9. ex BL conſtruitur. Sit *e.g.* BL ſectio conica & (*Fig.* 36.) BP erit per *Prop.* 11. circulus eodem centro diametro ipſo axi tranſverſo deſcriptus. Sit ſemiaxis tranſverſus $CB = a$, ſemiaxis conjugatus $= b$ eritque $CS = \sqrt{a^2 \pm b^2}$; ſit $SL = r$, $SP = x$, ſit SR ex puncto S in radium CP perpendicularis, & ſimilia erunt triangula SRP, SPL ob rectos angulos SPL, SRP & æquales SPR, PSL; proinde erit $PR = \frac{x^2}{r}$ & $CR = a \frac{\pm x^2}{r}$, $SR^2 = x^2 - \frac{x^4}{r^2}$, ſed $CS^2 = SR^2 + CR^2$ adeoque $a^2 \pm b^2 = \frac{a^2 r \mp 2ax^2 + x^2 r}{r}$, unde $x = b\sqrt{\frac{r}{2a - r}}$ ſi BL ſit Ellipſis & $x = b\sqrt{\frac{r}{2a + r}}$ ſi BL ſit Hyperbola; ſed vis centripeta eſt reciproce ut ſolidum $SP^2 \times SL \times \frac{lo}{Pp} =$ (cum area $SLl = \frac{1}{2} SL \times lo = \frac{1}{2} SP \times Ll$) $SP^3 \times \frac{Ll}{Pp} =$ (cum $Ll : Lo :: Pp : pn$) $SP^3 \times \frac{Lo}{Pn}$, ſed $Lo : Pn :: \overline{2a \pm r}^{\frac{3}{2}} \times r^{\frac{1}{2}} : ab$; adeoque vis centripeta erit reciproce ut $\frac{b^3 r^{\frac{3}{2}}}{\overline{2a \mp r}^{\frac{3}{2}}} \times \frac{\overline{2a \mp r}^{\frac{3}{2}} \times r^{\frac{1}{2}}}{ab} = \frac{b^2 r^2}{a}$ & proinde ob $\frac{b^2}{a}$ datam erit vis centripeta

reciproce

reciproce ut quadratum diſtantiæ SL. Si BL ſit Parabola circulus abit in tangentem ad verticem & vis centripeta erit reciproce ut quadratum diſtantiæ per *Corol.* 4. velocitas corporis ſectionem conicam deſcribentis eſt ut $\sqrt{\frac{2a \mp r}{r}}$ ſive in ſubduplicata ratione ſummæ vel differentiæ quantitatis quæ eſt reciproce ut diſtantia & datæ quantitatis.

Quod ſi detur vis centripetæ lex & quæratur curva quam corpus velocitate quavis projectum in medio non reſiſtente deſcribet, ea per *Prop.* 39. *Lib.* 1. *Princip.* indagari poteſt. Sed ſi vis centripeta ſit ut poteſtas aliqua *(m)* diſtantiæ curva innoteſcet *(Fig.* 37.*)* abſque illius Propoſitionis ope. Sit vis centripeta reciproce ut ſolidum a^{3-m} SL^m, ſitque SL$=r$, SP$=x$: Projiciatur corpus à puncto A ſecundum directionem AB, & ſit SB perpendicularis in AB, dicatur $\frac{1}{2SB^2} \pm \frac{1}{\overline{m-1} \times SA^2} = \frac{1}{c^2}$. His poſitis vis centripeta eſt in puncto L reciproce ut ſolidum $SP^3 \times \frac{Lo}{Pn} = \frac{x^3 \dot{r}}{\dot{x}}$, adeoque $a^{3-m} r^m = \frac{x^3 \dot{r}}{\dot{x}}$ & $\frac{\dot{x}}{x^3} = \frac{\dot{r}}{a^{3-m} r^m}$; quærantur harum Fluentes eritque $\frac{1}{2x^2} = \frac{1}{c^2} \pm \frac{a^{m-3}}{\overline{m-1} \times r^{m-1}}$, unde $x = c\sqrt{\frac{\overline{m-1} \times r^{m-1}}{\mp a^{m-3} c^2 + \overline{m-1} \times r^{m-1}}}$. Unde ſi vis centripeta ſit reciproce ut quadratum diſtantiæ adeoque $m=2$ erit SP $(=x) = c\sqrt{\frac{ar}{ar \mp c^2}}$ & ſi ST ſit perpendicularis in tangentem curvæ BP erit $ST = \frac{x^2}{r} = \frac{ac^2}{\mp c^2 + ar}$. Sed per *Prop.* 21. hujus *Partis* eſt momentum rectæ ST ad momentum rectæ SP ut SP ad PC radium curvaturæ Lineæ BP, *i. e.* $a\sqrt{ar} : c\sqrt{\mp c^2 + ar} :: c\sqrt{\frac{ar}{\mp c^2 + ar}} : PC = \frac{c^2}{a}$. Proinde radius curvaturæ BP eſt invariabilis, & Linea BP eſt circulus, adeoque per *Corol. Prop.* 11. Linea BL erit ſectio Conica. Ducatur BD conſtituens angulum DBS$=$ASB & in recta BD capiatur $BC = \frac{c^2}{a}$ eritque C centrum ſectionis conicæ S focus & $\frac{2c^2}{a}$ latus tranſverſum. Similis valor generalis perpendiculi SP inveſtigari poteſt ex *Prop.* 39. *Lib.* 1. *Princip.* Si enim ſit B altitudo loci à quo corpus cadens velocitatem acquireret æqualem ei qua corpus cur-

vam deſcribit in puncto L, erit ea velocitas per *Prop.* dictam & quadraturam Hyperbolæ altioris generis ut $\sqrt{\frac{a^{m-1} \mp r^{m-1}}{a^{m-1} r^{m-1}}}$; ſed velocitas in L eſt ſemper reciproce ut perpendiculum SP, adeoque erit SP ut $\sqrt{\frac{r^{m-1} \cdot a^{m-1}}{a^{m-1} \mp r^{m-1}}}$. Si corpus projiciatur ſecundum quamvis directionem velocitate quam acquireret cadendo ab infinita diſtantia ad altitudinem ſuam deſcribet curvam aliquam earum quas *Prop.* 15. conſideravimus; nam perpendiculum SP erit ut $r^{\frac{m-1}{2}}$ adeoque ut poteſtas aliqua radii.

Infinitæ aliæ excogitari poſſunt methodi particulares, quibus Lineæ aliquæ curvæ complexiores ex ſimplicioribus conſtrui poterunt præter eam quam hac *Prop.* & præcedenti ſectione fuſe tractavimus; ſiquidem vero omnibus immorari non liceat, ſufficiat nos eam adhibuiſſe tanquam reliquarum exemplum quæ inſigniora ſuppeditat theoremata & ſimul ad motus corporum curvilineos eruendos commodiſſima eſt. In rerum natura omnium corporum vires quibus in lineis curvis circumferuntur verſus unum idemque centrum diriguntur; atque ea ipſa Phænomena quæ per curvarum ordinatim applicatas ad datas axes explicari ſolent, ſi accurate examinentur, patebit ea ad unum centrum etiam referri debere. In natura nulla exiſtit axis ordinata aut abſciſſa, ſed corpus ad diſtantiam aliquam certa directione motus curvam deſcribens & centrum virium aliquod accurate vel quamproxime immotum. Ex velocitate (quæ ſemper eſt reciproce ut perpendiculum in directionem motus corporis) & ipſa motus directione, cujus menſura ex eodem perpendiculo eſt petenda, pendet natura curvæ data vi deſcripta, ſive lex vis centripetæ in data curva. Proinde methodus hæc curvas conſiderandi quam præcedentibus Propoſitionibus aliquot explicavimus ad Geometriam Philoſophiæ facilius applicandam commodiſſima eſt; cumque ſimul Lineæ curvæ aliquæ ordinum altiorum, quæ alia ratione non facile deſcribi poſſunt, ejus ope pulchre conſtruantur, ulteriori hac illuſtratione digna videbatur cum non ſatis luculenter explicata ſit in *Actis Philoſophicis* N° 359. Fatendum tamen rationem radii ad perpendiculum quam pro Lineæ curvæ æquatione uſurpavimus, non adeo plene curvam determinare in omni caſu ac ejuſdem æquationem ad axem poſitione datam; nam poſtquam hæc ratio indagata eſt ulteriori opus eſt calculo (niſi in caſibus ſpecialibus quorum aliquos ſupra tractavimus) ad curvæ indolem penitus dignoſcendam.

Ex *Prop.* 15. qua infinitas curvarum ſeries curvis duabus ejuſdem ſeriei atque rectis menſurari poſſe demonſtravimus, quadraturæ plures facili ratiocinio colligi poſſunt. Si enim curvæ cujuſvis ordinata ſit

$\frac{n}{a^{nm+1}}$

$\dfrac{a^{\frac{n}{nm+1}}}{\sqrt{a^{\frac{n}{nm+1}} - r^{\frac{n}{nm+1}}}}$ vel $\dfrac{r^{\frac{n}{nm+1}}}{\sqrt{r^{\frac{n}{nm+1}} - a^{\frac{n}{nm+1}}}}$ ſitque r abſciſſa, area hujus curvæ facile reduci poterit ad aream curvæ cujus ordinata $\dfrac{a^n}{\sqrt{a^n - r^n}}$ ſeu $\dfrac{r^n}{\sqrt{r^n - a^n}}$. Porro ſi BL ſit curva ejuſmodi (*Fig.* 38.) ut $LP = \dfrac{r^2 \sqrt{a^{2n} + r^n a^n}}{ar^n}$ ubi $SL = r$ & ex ea conſtruantur curvæ infinitæ ſumendo $BST : BSL :: m : 1$ & lineam SR in recta ST ad SB ut SL^m ad SB^m, ſitque $m = sn + 1$, & s ſemper numerus integer, eæ curvæ erunt omnes perfectæ rectificationis capaces. Sed hæc non ſunt hujus loci & curvarum quadraturæ Geometris adeo evaſerunt familiares ut his pluribus explicando operam perderem. Hujuſmodi etenim Problemata quæ non ita pridem ſummorum Geometrarum ingenia vehementer exercuerant facilia fere cujuſque oblectamenta reddita ſunt: Et ſane tanta ac tam ſubita hæc rei Geometricæ incrementa futuris ſæculis ſtupenda videbuntur niſi quodque ſuum reperiat *Neutonum.*

PROP. XXIII.

Deſcribat corpus curvam BL *in medio reſiſtente vi centripeta quavis data, exprimatur ratio vis centripetæ ad vim qua corpus curvam deſcriberet in vacuo quantitate z, eritque Reſiſtentia ut momentum hujus quantitatis ductum in arcum* Ll (Fig. 39.) *ſi area* BSL *uniformiter fluat.*

SIT Ll arcus tempore quovis t deſcriptus u velocitas corporis in puncto L, V vis centripeta qua corpus curvam in medio deſcribit, v vis centripeta qua eandem curvam in vacuo deſcriberet; ſit diſtantia $SL = r$, $SP = p$, arcus lo centro S radio Sl deſciptus $= x$, $Ll = y$, Reſiſtentia $= R$, Denſitas $= D$.

His poſitis ſi SP ſit perpendicularis in tangentem LP & ſi exponatur vis centripeta per rectam SL erit vis qua velocitas corporis acceleratur ex Gravitate oriunda ut LP, & proinde exponi poterit per $\frac{V\dot{r}}{\dot{y}}$; huic addatur reſiſtentia vel ex ea ſubducatur prout corpus aſcendit vel deſcendit in curva BL, & ſumma vel differentia erit vis qua velocitas corporis retardatur in aſcenſu vel acceleratur in deſcenſu corporis: Proinde ut $\frac{V\dot{r}}{\dot{y}} \pm R = \frac{-\dot{u}}{t} = \frac{-u\dot{u}}{\dot{y}}$ cum vis qua velocitas corporis acceleratur ſit ſemper ut velocitatis Fluxio directe & tempus quo fluit reciproce. Sed v eſt reciproce ut ſolidum $\frac{SL^2 \times lo^2}{lm} = \frac{p^2\dot{y}^2}{lm}$ & $V = \frac{lm}{t^2}$ unde $z = \frac{V}{v} = \frac{p^2\dot{y}^2}{t^2}$ & $u^2 = \frac{\dot{y}^2}{t^2} = \frac{z}{p^2}$ adeoque $-u\dot{u} = \frac{-\dot{z}}{2p^2} + \frac{z\dot{p}}{p^3} =$ (cum $\frac{\dot{p}}{p^3} = v\dot{r}$ per *Prop.* 21.) $\frac{-\dot{z}}{2p^2} + zv\dot{r} =$ (cum $p^2 = \frac{r^2\dot{x}^2}{\dot{y}^2}$ & $vz = V$) $\frac{-\dot{z}\dot{y}^2}{2r^2\dot{x}^2} + V\dot{r}$, unde $\frac{V\dot{r}}{\dot{y}} \pm R = \frac{-\dot{z}\dot{y}}{2r^2\dot{x}^2} + \frac{V\dot{r}}{\dot{y}}$ adeoque $R = \frac{\mp\dot{z}\dot{y}}{2r^2\dot{x}^2}$. Proinde ſi area BLS uniformiter fluit areola $LSl = \frac{1}{2}r\dot{x}$ dabitur & proinde reſiſtentia erit ut $\dot{z}\dot{y}$, *i.e.* ut momentum quantitatis z $(= \frac{V}{v})$ ductum in momentum curvæ.

Corol. I. Velocitas $u = \frac{\dot{y} \times \sqrt{z}}{r\dot{x}}$, unde ſi detur potentia velocitatis quæ cum denſitate compoſita rationem Reſiſtentiæ conſtituit dabitur medii denſitas; ſit Reſiſtentia ut denſitas & velocitatis quadratum eritque $D = \frac{\dot{z}}{2z \times \dot{y}}$.

Corol. II. Ex data denſitate medii, ejuſque gravitate acceleratrice ratio vis comprimentis ad denſitatem facile inveſtigari poteſt; quippe incrementum vis comprimentis oritur ex Gravitate incrementi Fluidi ſuperincumbentis adeoque erit ut incrementi illius Gravitas acceleratrix, moles & denſitas conjunctim; unde ſi vis comprimens dicatur F erit

F = D

$\dot{F} = DVr$; proinde ſi medium eadem vi acceleratrice ſupponatur urgeri qua corpus curvam deſcribens dabitur ratio vis comprimentis ad denſitatem. Hic obiter notari liceat ſi medii cujuſvis denſitas ſit compreſſioni proportionalis fore $\frac{\dot{D}}{D} = V\dot{r}$ & proinde ſi ordinatæ cujuſvis curvæ ſint ut vires centripetæ ad diſtantias curvæ abſciſſis æquales, erunt illius curvæ areæ denſitatum Logarithmi; adeo ut ſi vis centripeta ſit uniformis & $V = 1$ erit $\frac{\dot{D}}{D} = \dot{r}$, atque ipſæ diſtantiæ erunt denſitatum Logarithmi ut olim demonſtravit Cl. *Halleyus* eximius Geometriæ Profeſſor *Savilianus*. Quod ſi curvæ cujuſvis ordinatæ ſint ut denſitates, erunt vires centripetæ reciproce ut ſubtangentes ejuſdem curvæ cum $\frac{D\dot{r}}{\dot{D}} = \frac{1}{V}$. Atque hujus *Corollarii* ope *Prop.* 21 & 22. *Lib.* 2. *Princip.* cum hujus *Scholio* facile demonſtrari poſſunt.

PROP. XXIV.

Deſcribat corpus curvam aliquam Prop. 14. *quarum perpendicula* SP *ſunt in ratione poteſtatis* $(n+1)$ *radiorum, vi centripeta quæ ſit reciproce ut poteſtas* (m) *diſtantiæ, eritque medii denſitas (ſi angulus* LST *ſit rectus) reciproce ut tangens* LT, *Reſiſtentia ad gravitatem ut* $\overline{2n-m+3} \times LP$ *ad* $\overline{2n+2} \times SL$ *& velocitas ad velocitatem corporis ad eandem diſtantiam circulum deſcribentis in vacuo ut* 1 *ad* $\sqrt{n+1}$.

MAnifeſtum eſt ex *Prop.* 22. eſſe $v = \frac{\overline{n+1} \times a^{2n}}{r^{2n+3}}$ adeoque erit $V : v :: r^{2n-m+3} : \overline{n+1} \times a^{2n-m+3}$ unde $z = \frac{V}{v} = \frac{r^{2n-m+3}}{\overline{n+1} \times a^{2n-m+3}}$, & R =

& $R = \frac{\mp L l \times \dot{z}}{2r^2 \dot{x}^2} =$ (cum $Ll = \dot{y}$) $\frac{\mp \overline{2n - m + 3} \times r^{2n-m} \dot{r} \dot{y}}{\overline{2n+2} \times a^{2n-m+3} \dot{x}^2}$, ſed Ll $(= \dot{y})$: lo $(= \dot{x})$ :: SL $(= r)$: $SP = \frac{r^{n+1}}{a^n}$, atque Lo $(= \dot{r})$: lo $(= \dot{x})$ :: LP : $SP = \frac{r^{n+1}}{a^n}$, unde $R = \frac{\mp \overline{2n - m + 3} \times r^{-m} \times LP}{\overline{2n+2} \times a^{-m+3} \times SL}$, ſed $V = \frac{r^{-m}}{a^{-m+3}}$, & proinde $R : V :: \mp \overline{2n - m + 3} \times LP : \overline{2n+2} \times SL$.

Velocitas eſt æqualis $\frac{\dot{y}\sqrt{\dot{z}}}{r\dot{x}} = \frac{a^{\frac{m-3}{2}}}{\sqrt{n+1} \times r^{\frac{m-1}{2}}}$. Sed velocitas corporis circulum deſcribentis in vacuo ad diſtantiam SL vi centripeta $\frac{a^{m-3}}{r^m}$ eſt $\frac{a^{\frac{m-3}{2}}}{r^{\frac{m-1}{2}}}$, per *Prop.* 4. *Lib.* 1. *Princ.* Proinde velocitas eſt ad velocitatem corporis circulum ad eandem diſtantiam in vacuo deſcribentis ut 1 ad $\sqrt{n+1}$.

Denſitas eſt directe ut reſiſtentia & reciproce ut velocitatis quadratum adeoque erit $D = \frac{\mp \overline{2n - m + 3} \times LP}{\overline{2n+2} \times SL^2}$ ſed LT : SL :: SL : LP, adeoque $D = \frac{\mp \overline{2n - m + 3}}{\overline{2n+2} \times LT}$; proinde denſitas eſt reciproce ut tangens LT.

Corol. I. Si BL ſit curva quæ deſcribi poſſet vi centripeta diſtantiæ cuivis potentiæ proportionali, erit univerſaliter denſitas medii reciproce ut tangens LT; quippe licet SP non ſit ut poteſtas aliqua radii SL ſi $v = \frac{a^{s-3}}{r^s}$, ſive vis centripeta corporis curvam in vacuo deſcribentis ſit reciproce ut potentia quævis s diſtantiæ, erit $z = \frac{V}{v} = \frac{r^{s-m}}{a^{s-m}}$ & $R = \frac{\mp \overline{s - m} \times r^{s-m-3} \dot{r} \dot{y}}{a^{s-m} \dot{x}^2}$, & velocitas $u = \frac{\dot{y}^2}{r^2 \dot{x}^2} \times \frac{r^{s-m}}{a^{s-m}}$, unde $D = \frac{R}{u^2}$

=

$= \frac{\mp \dot{r} \times \overline{s-m}}{\dot{y} r}$, ſed L$o$ ($= \dot{r}$) : Ll ($= \dot{y}$) : SL ($= r$) : LT $= \frac{r\dot{y}}{\dot{r}}$, adeoque $D = \frac{\pm \overline{s-m}}{LT}$, *i.e.* Denſitas erit in omnibus iis curvis reciproce ut tangens LT. Atque hinc Denſitas medii in quo corpus ſectionem quamvis conicam deſcribit vi quavis centripeta verſus centrum vel focum tendente innoteſcit.

Corol. II. In curvis *Prop.* 14. quarum $SP = \frac{r^{n+1}}{a^n}$ eſt SL : LP :: a^n : $\sqrt{a^{2n} - r^{2n}}$ unde $R = \mp \frac{\overline{2n+3-m} \times \sqrt{a^{2n} - r^{2n}}}{\overline{2n+2}\, a^{n+3-m}\, r^m}$ ſive ut $\frac{\sqrt{a^{2n} - {}^{2n}}}{r^m}$ & Denſitas erit ut $\frac{\sqrt{a^{2n} - r^{2n}}}{r}$ & velocitas reciproce ut $\frac{m-1}{2}$ poteſtas diſtantiæ.

Corol. III. Cum ſit R : V :: $\overline{2n+3-m} \times LP$: $\overline{2n+2} \times SL$ ut corpus in harum curvarum aliqua moveatur oportet quantitatem $\overline{2n+2} \times R$ minorem eſſe $\overline{2n+3-m} \times V$; ſi enim æquales ſint erit LP = SL & Lo = Ll adeoque corpus recta deſcendet ad centrum.

Corol. IV. Si Reſiſtentia ſit in ratione Denſitatis & Velocitatis ſeu $R = Du$ erit Denſitas ut $\frac{\sqrt{a^{2n} - r^{2n}}}{r^{\frac{m+1}{2}}}$.

Corol. V. Sit circulus *Exemplum* 1 (*Fig.* 40.) ſitque centrum virium in circumferentia, atque erit $n = 1$ cum ſit SL : SP :: SB ($= a$) : SL ($= r$). Denſitas, quæ eſt reciproce ut LT ſive $\frac{SL^2}{LP}$, erit ut $\frac{LB}{SL}$ cum ſit LP : SL :: BL : SB. Proinde Denſitas erit ſemper directe ut diſtantia corporis a Puncto B Centro virium oppoſito, & reciproce ut diſtantia ab ipſo centro S. Velocitas erit ad velocitatem corporis circulum ad eandem diſtantiam in vacuo deſcribentis ut 1 ad $\sqrt{2}$. Reſiſtentia $R = \mp \frac{\overline{s-m} \times a^{m-4} \times LB}{4\, SL^m}$ ſive ut $\frac{LB}{SL^m}$; eritque ad Gravitatem ut $\mp \overline{s-m} \times BL$ ad 4BS; & denſitas ſi $R = Du$ erit ut $\frac{LB}{SL^{\frac{m+1}{2}}}$. Hinc

1°. Si vis centripeta ſit reciproce ut quadratum diſtantiæ, erit reſiſtentia ad

ad Gravitatem ut 3BL ad 4SB ſive ut 3L*o* ad 4L*l*, adeoque erit 4R ſemper minor 3V; ſed ſi 4R = 3V erit L*o* = L*l* & corpus directe deſcendet verſus centrum velocitate cujus dupla ea erit quam corpus acquireret caſu à diſtantia infinita; ut ex *Theor.* 10. *Lib.* 1. *Princip.* facile colligitur; adeoque tempus quo corpus cadit à B ad S in medio erit ad tempus quo corpus poſt caſum infinitum in vacuo eandem altitudinem deſcriberet ut 1 ad $\sqrt{2}$. 2. Reſiſtentia & Denſitas in puncto B evaneſcunt; nam Denſitas eſt ſemper ut $\frac{LB}{SL}$. Proinde corpus unicum tantum circulum in dato medio deſcribere poteſt, dato enim medio dabitur punctum B in quo ejus Denſitas eſt nulla adeoque circulus BSL.

Corol. VI. Sit BL Spiralis Logarithmica (*Fig.* 41.) quæ in eodem angulo Radiis ſemper occurrit eritque SL ad SP in data ratione Radii curvaturæ LC ad Radium SL; adeoque ſi LC : SL :: *a* : *b* erit SP $= \frac{SL \times b}{a}$ & $n = o$; proinde Denſitas medii, quæ eſt reciproce ut tangens LT $= \frac{SL^2}{LP} =$ (cum SL : LP :: LC : CS :: *a* : $\sqrt{a^2 - b^2}$) $\frac{SL \times a}{\sqrt{a^2 - b^2}}$, erit ſemper reciproce ut diſtantia SL à centro virium; quæcunque ſit virium Centripetarum Lex, ſi modo ſit reciproce ut potentia aliqua diſtantiæ. Reſiſtentia erit ſemper in data ratione ad vim Centripetam, nam $R = \frac{\mp \overline{3 - m}}{2a^{3-m}\, SL^m} \times \frac{CS}{LC}$ ſed $V = \frac{a^{m-3}}{r^m}$, adeoque R : V :: $\mp \overline{3 - m} \times CS : 2CL$, & proinde in data ratione; cum ex natura Spiralis detur angulus CLS & ratio CS ad CL. Velocitas eſt ad velocitatem corporis ad eandem diſtantiam circulum in vacuo deſcribentis ut 1 ad $\sqrt{n+1}$, *i.e.* ut 1 ad 1; proinde velocitas corporis ea ipſa eſt qua circulum in vacuo ad eandem altitudinem deſcriberet & ad eam quam caſu infinito acquireret ad eam altitudinem ut 1 ad $\sqrt{2}$. Hinc 1 ſi Reſiſtentia ſit in ratione Denſitatis & Velocitatis, & vis centripeta directe ut diſtantia, erit Fluidum Homogeneum. Quippe in hoc caſu eſt $m = -1$, $R = \frac{2SL}{a^4} \times \frac{CS}{LC}$ & $u = \frac{SL}{a^2}$, adeoque $D = \frac{R}{u} = \frac{2CS}{a^2 \times LC}$, proinde erit invariata ob datam rationem CS ad LC. Unde ſi corporum motui reſiſtatur in ratione velocitatis & corpus quodvis in oceano noſtro projiciatur velocitate æquali ei qua circulum in vacuo ad eandem diſtantiam deſcriberet; Corpus Spiralem hanc æquiangulum percurret cujus centrum erit in centro terræ ſi modo Reſiſtentia medii ab initio motus

tus ſit major duplo Gravitatis corporis. 2. Sit vis centripeta reciproce ut diſtantia & quæcunque ſit curva ſi $SP = \frac{SL^{n+1}}{a^n}$ erit Reſiſtentia ad Gravitatem LP ad SL; proinde Reſiſtentia ſemper debet eſſe minor vi Gravitatis. Velocitas eſt ſemper uniformis; quippe $U = \frac{a^{\frac{m-3}{2}}}{\sqrt{n+1} \times r^{\frac{m-1}{2}}} = \frac{1}{\sqrt{n+1} \times a}$. Denſitas eſt ut Reſiſtentia adeoque reciproce ut diſtantia ſive $R = DU$ ſive $R = DU^2$; hæc univerſaliter obtinent in omni caſu quo V eſt reciproce ut diſtantia & v reciproce ut potentia quævis diſtantiæ. 3. Sit vis centripeta reciproce ut quadratum diſtantiæ, & reſiſtentia erit ad vim centripetam ut CS ad 2CL quemadmodum olim demonſtravit D. *Neutonus*, *Prop.* 11. *Lib.* 2. *Princip.* Si $R = Du$ erit Denſitas in reciproca ſeſquiplicata ratione diſtantiæ; ſi vero $R = Du^2$ erit Denſitas, ut prius univerſaliter oſtendimus, reciproce ut diſtantia; vis qua Fluidum comprimitur ſive pondus Fluidi incumbentis erit ut Denſitatis quadratum cum $\dot{F} = DV\dot{r} = \frac{\dot{r}}{r^3}$, adeoque F reciproce ut r^2: Ut corpus curvam deſcribat neceſſe eſt vim reſiſtentiæ minorem eſſe dimidio vis centripetæ; ſi enim $2R = V$ erit $CS = CL$, adeoque Spiralis coincidet cum SL, & corpus recta deſcendet ad centrum velocitate quæ erit ad velocitatem caſu infinito ad eandem altitudinem in vacuo acquiſitam ut 1 ad $\sqrt{2}$, & tempora deſcenſus erunt in eadem ratione reciproca: Specifica hujus Fluidi Gravitas eſt reciproce ut Cubus diſtantiæ.

Corol. VII. Sit BL Epicyclois deſcripta motu puncti in circumferentia circuli circa alium æqualem revolventis poſiti; ſitque S ubi punctum deſcribens tangit baſim, eritque $n = \frac{1}{2}$, adeoque reſiſtentia $= \frac{\overline{4-m} \times LP \times a^{m-3}}{3 \times SL^{m+1}}$, & proinde ut $\sqrt{SB - SL} \times SL^{-m}$. Velocitas eſt ad velocitatem corporis circulum in vacuo ad eandem diſtantiam deſcribentis ut $\sqrt{2}$ ad $\sqrt{3}$. Denſitas ſi $R = Du^2$ erit ut $\frac{\sqrt{SB - SL}}{SL}$ & ſi $R = DU$ erit D ut $\sqrt{\frac{SB - SL}{SL^{m+1}}}$. Reſiſtentia eſt ad Gravitatem ſemper ut $\mp \overline{4-m}$ LP ad 3SL, ſive ut $\mp \overline{4-m} \times BN$ ad 3SB, & proinde 3R ſemper minor erit $\mp \overline{4-m} \times V$; ſi enim æquales ſint corpus recta deſcendet ad centrum velocitate quæ ſemper erit ad velocitatem corporis circulum

 in

in vacuo deſcribentis ad eandem diſtantiam ut $\sqrt{2}$ ad $\sqrt{3}$. Reſiſtentia & Denſitas medii evaneſcunt in B.

Corol. VIII. Sit BL Parabola, & Reſiſtentia erit ad vim centripetam ut $\mp \overline{2-m} \times \sqrt{SB-SL}$ ad $\sqrt{SL}$; velocitas erit ſemper ad velocitatem corporis circulum ad eandem diſtantiam in vacuo deſcribentis ut $\sqrt{2}$ ad 1. Denſitas ſi $R = Du^2$ erit ut $\sqrt{\frac{SL-SB}{SL^{\frac{3}{2}}}}$ & ſi $R = Du$ erit D ut $\sqrt{\frac{SL-SB}{SL^{m+2}}}$. Proinde Denſitas & Reſiſtentia in vertice Parabolæ evaneſcunt.

Corol. IX. Si corpus deſcendat erit ſemper vis centripeta ad reſiſtentiam ut $\overline{2n+1} \times SL$ ad $\overline{2n+3-m} \times LP$, unde ſi $2n+3$ non ſit major m erit reſiſtentia imaginaria. Si corpus aſcendat erit R : V :: $\overline{m-3-2n} \times LP$ ad $\overline{2n+2} \times SL$, adeoque ſi $2n+3$ ſit major m erit reſiſtentia in hoc caſu imaginaria.

Corol. X. Si $m = 2n+3$, *i. e.* ſi vis centripeta ſit reciproce ut potentia $2n+3$ diſtantiæ erit reſiſtentia nulla.

Schol. Æquali fere facilitate reſiſtentia inveſtigatur in aliis curvis ex *Prop.* 23. ſit BL (*Fig.* 22. *n.* 2.) Spiralis Hyperbolica in qua LP : SP :: SL : SB, per *Corol.* 1. *Lemmatis* 2. adeoque $SP = \frac{ar}{\sqrt{a^2+r^2}}$ & $\frac{1}{v} = \frac{Lo \times SP^3}{Pn} = \frac{\overline{a^2+r^2}^{\frac{3}{2}}}{a^1} \times SP^3 = r^3$. In hac curva erit $\frac{V}{v} = z = \frac{a^{m-3}}{r^{m-3}}$ & proinde $R = \frac{\mp \overline{3-m} \times a^{m-3}}{2r^{m-1}} \times \frac{\sqrt{r^2+a^2}}{a}$, adeoque ut $r^{1-m} \times \sqrt{r^2+a^2}$; velocitas eſt ut $r^{\frac{1-m}{2}} \times \sqrt{a^2+r^2}$, & Denſitas ſi $R = Du$, eſt reciproce ut $\frac{m-1}{2}$ poteſtas diſtantiæ, ſed ſi $R = Du^2$ erit Denſitas reciproce ut $\sqrt{a^2+r^2}$. Hinc ſi vis centripeta ſit ut diſtantia erit Denſitas (ſi $R = Du$) ut diſtantia, ſi vis centripeta ſit reciproce ut diſtantia erit Fluidum homogeneum; ſit vis centripeta reciproce ut Quadratocubus diſtantiæ, eritque $\frac{m-1}{2} = 2$, adeoque Denſitas erit reciproce ut quadratum diſtantiæ ſi corpus aſcendat in Spirali. Theorema generale *Prop.* 23. qua Reſiſtentia & Denſitas medii ex data curva & vi centripeta facile

 inveſtigari

inveſtigari poſſunt ante aliquot annos inveni priuſquam Tractatus Cl. *Bernoailli* inclyti Matheſeos Profeſſoris *Baſilienſis* in Commentariis Academiæ *Pariſienſis* de eadem materia evulgati ad manus pervenerant; nec quidem difficile fuerat hæc ex ipſis *Princip. Neutoni*, *Lib.* 2. *Prop.* 10 & 11. eruere. Cum vero Cel. Viri qui materiam hanc ſoli tractarunt motus corporum in Spirali tantum logarithmica conſideraverint; idem in aliis aliquibus curvis ſimpliciſſimi hujus Theorematis ope indagare videbatur operæ prætium.

PROP. XXIV. PROBLEMA.

Deſcribat corpus curvam quamvis datam AMB *(Fig.* 44.*) vi centripeta uniformi tendente ad centrum maxime longinquum invenire velocitatem corporis, Reſiſtentiam ac Denſitatem medii.*

SIT AE axis curvæ Horizonti parallela, & vis centripeta agat ſecundum directionem ordinatarum PM, *pm*. Fluat area APM uniformiter, ſitque PM $= r$, AP $= x$, Pp $=$ mQ $= \dot r$, Mm $= \dot y$ eritque mR $= \ddot r$, & vis centripeta v qua corpus curvam deſcriberet in vacuo erit ut $\frac{\ddot r}{\dot x^2}$, adeoque $z = \frac{V}{v} = \frac{\dot x^2}{\ddot r}$, cum V $= 1$; adeoque Reſiſtentia quæ eſt ut $\dot z \dot y$ erit æqualis $\frac{\mp \dot y \dddot r}{2 \ddot r^2}$. Velocitas corporis eſt ut $\frac{\dot y}{\sqrt{\ddot r}}$; & proinde ſi R $=$ Du^2 erit D $= \frac{\dddot r}{2 \ddot r \dot y}$. Reſiſtentia vero erit ad vim centripetam ut $\dot y \times \dddot r$ ad $\ddot r^2$.

Corol. I. Hinc ſi AMB ſit circulus Centro C Diametro AB deſcriptus. Sitque CP $= x$, PM $= r$; erit $r^2 = a^2 - x^2$ unde $\ddot r = \frac{a^2 - \dot x^2}{\overline{a^2 - x^2}^{\frac{3}{2}}}$,

$\dddot r =$

$\dddot{r} = +\frac{3a^2 x\dot{x}^3}{\overline{a^2 - x^2}|^{\frac{5}{2}}}$ & $\dot{y} = \frac{a\dot{x}}{\sqrt{a^2 - x^2}}$, & proinde $R = \frac{3x}{2a}$ & Reſiſtentia ad Gravitatem ut 3CP ad 2CB; velocitas eſt ut $\frac{\dot{y}}{\sqrt{\ddot{r}}}$ adeoque in ſubduplicata ratione ipſius r; Denſitas ſi $R = Du^2$ eſt ut $\frac{3ax}{2ay}$ adeoque (ob triangula ſimilia MPC, MCT) ut tangens MT; corpus in hoc circulo deſcendere poteſt à D ad A ſed ex A ad D hac vi centripeta aſcendere nequit.

Corol. II. Deſcribat corpus Parabolam vel Hyperbolam *(Fig.* 44.*)* cujuſcunque dignitatis, ſitque $x = r^m$, & ſi ordinatæ r parallelæ ſint directioni vis centripetæ erit Denſitas reciproce in ſubduplicata ratione ipſius r, ſi $R = Du$; ſed ſi $R = Du^2$ erit Denſitas reciproce ut tangens terminata puncto contactus M & axi vel Aſſymptoto parallela Horizonti.

Corol. III. Sit AMB Logiſtica *(Fig.* 47.*)* ſitque Aſſymptotos perpendicularis Horizonti & $\frac{\dot{r}}{a} = \frac{\dot{x}}{x}$ ex natura curvæ, adeoque $\dddot{r} = \frac{2a\dot{x}^3}{x^3}$, $\ddot{r} = \frac{a\dot{x}^2}{x^2}$ & $\dot{y} = \frac{\dot{x}}{x}\sqrt{a^2 + x^2}$. Proinde $R = \frac{\sqrt{a^2 + x^2}}{a}$ adeoque reſiſtentia eſt ut tangens MT. Velocitas eſt ut $\frac{\dot{y}}{\sqrt{\ddot{r}}}$ adeoque ut tangens MT; & proinde Denſitas erit conſtans & fluidum homogeneum ſi $R = Du$; ſed ſi $R = Du^2$ erit Denſitas reciproce ut tangens MT: Corpus igitur in Fluido homogeneo projectum velocitate, cujus pars verticalis eſt æqualis terminali corporis cadentis, deſcribet curvam Logarithmicam ſi reſiſtentia ſit in ratione velocitatis quemadmodum olim aſſeruit Cel. *Hugenius* ad finem Tractatus de Cauſa Gravitatis.

Corol. IV. Deſcribat corpus curvam cujus conſtructionem ex Hyperbola deduxit D. *Neutonus*, *Prop.* 4. *Lib.* 2. *Princip.* Sit $R = r$ *(v. Fig.* apud *Neutonum)* $CR = x$, $CD = a$, $DG = b$, ordinatæ Hyperbolæ $TR = z$, $DE = c$, cumque Rr $(= r)$ $= \frac{GELT}{N}$ erit $\dot{r} = \frac{ab - cx}{Nx}\dot{x}$, $\ddot{r} = \frac{-ab\dot{x}^2}{Nx^2}$ $\dddot{r} = \frac{+2ab\dot{x}^3}{Nx^3}$, & proinde $R = \frac{N^2 x\dot{y}}{ab\ \dot{x}} = \frac{N^2}{ab} \times rL$. Proinde

Reſiſtenti

ſiſtentia eſt ut tangens curvæ; velocitas eſt ut $\frac{\dot{y}}{\sqrt{\dot{r}}} = \sqrt{\frac{N}{ab}} \times \frac{\dot{xy}}{\dot{x}}$ adeoque in eadem ratione: Unde ſi R = D*u* erit Denſitas invariata & fluidum homogeneum; & proinde curva ea eſt quam projectile in medio ſimilari deſcribere poſſet vi centripeta uniformi tendente ad centrum infinite diſtans ſi motui corporis reſiſteretur in ratione velocitatis.

Ex hac *Prop.* demonſtrari poſſunt omnia quæ ſerierum ope deduxit D. *Neutonus Prop.* 10. ejuſdem Libri. Ex *Prop.* 23. inveniri poteſt Regula generalis qua Reſiſtentia & Denſitas medii inveſtigari poſſunt cum corpus curvam deſcribit vi quavis centripeta non uniformi tendente ad centrum maxime longinquum; ſed his non diutius immorabimur.

Hac Sectione duo Problemata inſigniora Philoſophiæ Mathematicæ tractavimus eo præſertim conſilio ut appareat uſus Linearum curvarum in Philoſophia Naturali; neque hæc à propoſito noſtro aliena habuimus quod his explicentur methodi Lineas curvas deſcribendi elegantiſſimæ utpote quæ ab ipſa natura uſurpantur, nec arte quavis videntur poſſe imitari; Quodque hinc ſimpliciores derivari poſſint Ideæ deſcriptionis curvarum aliquarum admodum complexarum quam ex alio quovis fonte.

SECTIO V.

De Deſcriptione Linearum Geometricarum per data Puncta.

LEMMA III.

Linea Ordinis (n) occurrere poteſt Lineæ ſecundi Ordinis in punctis duntaxat (2n) & Lineæ tertii Ordinis in Punctis tantum 3n.

SIT $y^n + \overline{ax+b} \times y^{n-1} + \overline{cx^2+dx+e} \times y^{n-2}$, &c. $+ \overline{kx^{n-1} + lx^{n-2}}$ &c. $\times y + mx^n + px^{n-1}$, &c. $= o$. Æquatio generalis ad Lineam Ordinis (*n*) ubi *x*, *y* deſignant Lineæ abſciſſam & ordinatam *a*, *b*, *d*, *c*, &c. datas denotant quantitates. Sit $y^2 + \overline{Ax + B} \times y + Cx^2 + Dx + E = o$ æquatio generalis ad Lineam ſecundi Ordinis. In prima æquatione pro y^2 ſubſtituatur valor ejus quæ ex ſecunda conſequitur; pro

pro y^n, y^{n-1}, y^{n-2} ſimiliter ſubſtituantur in æquatione ad Lineam Ordinis (n) valores earum quæ ex ſecunda deduci poſſunt & hujuſmodi operatione prodibit æquatio in qua erit y unius dimenſionis; unde valor eruetur ipſius y hujus formæ $y = \frac{Kx^n + Lx^{n-1}, \&c.}{Hx^{n-1} + Gx^{n-2}, \&c.}$ Loco y in æquatione ad Lineam ſecundi Ordinis ſubſtituatur hic valor & in æquatione reſultante abſciſſa x erit dimenſionis $2n$. Sed ubi Linea Ordinis (n) & Linea Ordinis ſecundi ſibi mutuo occurrunt earum abſciſſæ & ordinatæ ad eandem axem fiunt æquales; & proinde æquatio hæc oſtendit valores abſciſſæ varios eſſe $2n$, quibus Lineæ dictæ ſibi mutuo occurrere poſſunt; ſed non poſſe eſſe plures. Unde harum curvarum occurſus poſſunt eſſe $2n$ & nunquam plures.

Sit $y^3 + \overline{Ax + B} \times y^2 + \overline{Cx^2 + Dx + E} \times y + fx^3 + gx^2 + bx + i = 0$. Æquatio generalis ad Lineam tertii Ordinis: Et in æquatione ad Lineam Ordinis (n) ſcribe pro y^n, y^{n-1}, y^{n-2}, y^2, &c. Valores quæ ex hac conſequuntur & exterminabuntur omnes dimenſiones ordinatæ y præter y^2 & y; in hujus coefficiente erit x dimenſionis $n-1$, in coefficiente illius dimenſionis $n-2$. Sint m, p, r quantitates quæ componuntur ex dimenſionibus $n-2$, $n-1$, n abſciſſæ & omnibus inferioribus. Sintque i, b, k coefficientes terminarum y^2, y, y^0 in æquatione propoſita ad Lineam tertii Ordinis; eritque 1. $y^3 + iy^2 + by - k = 0$. 2. $my^2 + py - r = 0$. Ducatur prima in m & ſecunda in y eorumque differentia dabit $y = \frac{pkm - r^2 - bmr}{\overline{mi-p} \times r - m^2k}$. In æquatione ſecunda pro y ſcribe hunc ejus valorem & dein æquationem reducendo prodibit $mi^2r^2 - ipr^2 - 2krm^2i + k^2m^5 - p^3k \times \overline{mi-p} + prb \times \overline{mi-p} - pkm^2b + pkrm - r \times \overline{bm + r}^2 = 0$. Sed in valoribus quantitatum m, p, r, k, b, e abſciſſa x erat dimenſionum $n-2$, $n-1$, n, 3, 2, 1 reſpective; adeoque ſi ſubſtituantur earum valores, æquatio prodibit dimenſionis $3n$ abſciſſæ x ſed non plurium. Proinde concurſus Lineæ Ordinis (n) & Lineæ tertii Ordinis poſſunt eſſe $3n$ & nunquam plures.

Corol. I. Hinc Linearum Ordinum m & n occurſus videntur eſſe mn. Si altera ſit Linea Parabolici generis manifeſtum eſt earum occurſus fore mn; ſit enim $y = x^m$ & $y^n = x^{mn}$ unde exterminando y ex altera æquatione evadet x dimenſionum mn. Cumque occurſus plures nunquam eſſe poſſint Linearum Ordinis (n) atque Ordinis vel ſecundi vel tertii regula univerſaliter ad altiores ordines extendi poſſe videtur. Omnes altiorum curvarum caſus quos adhuc tentavimus idem probant; univerſalem vero hujus rei demonſtrationem adhuc fruſtra quæſivimus propterea quod difficile ſit diviſores in arduis æquationibus invenire.

Corol. II

Corol. II. Linea Ordinis (n) occurrere poteſt aliæ ejuſdem Ordinis in punctis n^2. Proinde duæ Lineæ Ordinis (n) per eadem puncta n^2 tranſire nonnunquam poſſunt; adeoque puncta data quorum numerus eſt $\frac{1}{2}\overline{n^2+3n}$ non ſufficiunt ad Lineam Ordinis (n) ita determinandam ut unica ſit curva quæ per ea data puncta duci poſſit: Cum vero coefficientes in æquatione generali ad Lineam Ordinis (n) ſint $\frac{1}{2}\overline{n^2+3n}$, patet ſi plura dentur puncta, Lineam Ordinis (n) per ea forſan duci non poſſe & Problema reddi poſſe impoſſibile. Sic novem puncta non adeo plene determinant Lineam Ordinis tertii ac quinque Lineam Ordinis ſecundi, decem tamen ad Lineam tertii Ordinis determinandam nimia ſunt.

Corol. III. Hinc ſi punctorum quæ ad Lineam Ordinis (n) determinandam proponuntur ſint $nr+1$ in Linea Ordinis inferioris (r) Problema vel erit impoſſibile vel conficitur ope Lineæ illius Ordinis (r) & alterius Lineæ Ordinis $(n-r)$ per puncta n^2-r-1 deſcriptæ. Si enim Linea Ordinis (n) per data puncta duci poſſet concurſus ejus Lineæ & alterius Ordinis (r) poſſent eſſe $nr+1$; qui plures nr eſſe nequeunt.

Corol. IV. Linea Ordinis (n) nequit habere puncta duplicia plura $\frac{1}{2}\times\overline{n^2-3n+2}$. Quippe Linea Ordinis $(n-2)$ duci poteſt per puncta $\frac{1}{2}\overline{n^2-3n+2}+n-2$; unde ſi Lineæ Ordinis (n) puncta $\frac{1}{2}\overline{n^2-3n+2}$, $+1$ duplicia eſſent; Linea Ordinis (n) occurreret Lineæ Ordinis $(n-2)$ per ea duplicia & alia ſimplicia $n-3$ deſcriptæ in punctis n^2-2n+1; earum vero occurſus nequeunt eſſe plures n^2-2n. Linearum igitur Ordinum 3, 4, 5, &c. puncta duplicia non poſſunt eſſe plura 1, 3, 6, &c.

Corol. V. Si in tribus Lineæ Ordinis (n) punctis concurrant arcus $(\frac{1}{2}n)$ & in quarto arcus $(\frac{1}{2}n-1)$ reliqua omnia puncta Lineæ erunt ſimplicia.

PROP. XXV. PROBLEMA.

Lineam Ordinis (n) ducere per data puncta $2n+1$, in quorum aliquo oportet arcus curvæ $(n-1)$ concurrere.

IMprimis data puncta Lineam ita determinare ut unica ſit curva quæ Problemati ſatisfacere poteſt hinc patet; quod ſi duæ Lineæ Ordinis (n) per eadem puncta $2n+1$ tranſirent, & in eodem puncto nodum arcuum curvæ $n-1$ ſe mutuo decuſſantium haberent, concurſus harum Linearum forent n^2+1; qui plures n^2 eſſe nequeunt. Linea quarti Ordinis ducitur per novem puncta, quorum unicum eſt triplex, eadem ratione ope Lineæ tertii Ordinis, Nodum ibidem habentis, qua Linea ea tertii Ordinis

ducitur per ſeptem puncta ope ſectionis conicæ ſecundum *Corol. 9. Prop. 1.* hujus Partis. Sint C, S, B (*Fig.* 47.) tria datorum punctorum ſitque S punctum triplex; jungantur CB, CS, BS & circa Polos C, S rotentur anguli BCS, BSC. Crurum CB, SB concurſus applicetur ſucceſſive reliquis ſex punctis per quæ Linea eſt ducenda; & interea notentur ſex puncta in quibus crura CS, SC ſe mutuo decuſſant; per hæc ſex puncta atque Polum S ducatur Linea tertii Ordinis punctum duplex habens in S; & ſi per eam ducatur concurſus crurum CS, SC concurſus crurum CB, SB deſcribet Lineam quarti Ordinis punctum triplex habentem in Polo S tranſeuntem per reliqua octo data puncta. Similiter ope Lineæ quarti Ordinis punctum triplex habentis, ducenda eſt Linea quinti Ordinis, puncto quadruplice donata, per undecim puncta; & univerſaliter Linea hujuſmodi Ordinis (*n*) ducenda eſt ope alterius Ordinis (*n*—1) punctum multiplex habentis arcuum (*n*—2) concurrentium in eodem Polo; hæc vero deſcribenda eſt ope alterius Ordinis (*n*—2) & ſic deinceps donec ad ſectiones Coni perveniatur quæ rectarum ope deſcribi poſſunt.

PROP. XXVI. PROBLEMA.

Ducere Lineam Ordinis (2n) per puncta quot ſufficiunt ad determinandam Lineam Ordinis (n) & tria alia quæ ſingula nodi ſunt arcuum curvæ (n) ſe mutuo decuſſantium.

SInt C, S, B (*Fig.* 47.) tria puncta multiplicia data; jungantur CS, CB, SB & circa Polos C, S rotentur anguli BCS, BSC; crurum CB, SB concurſus applicetur ſucceſſive punctis datis quæ ſufficiunt ad determinandam Lineam Ordinis (*n*) & notentur puncta in quibus reliqua interea crura ſe mutuo decuſſant; per hæc puncta dein ducatur Linea Ordinis (*n*) & ſi concurſus crurum CS, SC per eam ducatur, concurſus crurum CB, SB deſcribet Lineam Ordinis 2*n* per data puncta tranſeuntem, per *Prop.* 3. hujus Partis. Plures nonnunquam erunt Lineæ quæ huic Problemati ſatisfacere poterint.

PROP.

Prop. XXVII. Problema.

Ducere Lineam Ordinis (2n) per puncta data 2n + 4, quorum tria ſunt nodi arcuum curvæ (n) in iiſdem punctis concurrentium, & quartum ſimilis nodus arcuum (n — 1).

Sint ut prius C, S, B (*Fig.* 48.) puncta in quibus arcus concurrere debent quorum numerus (*n*) ſit A punctum ubi concurrunt arcus (*n* — 1) jungantur CS, CB, SB; & circa Polos C, S rotentur anguli BCS, BSC; applicetur concurſus crurum CB, SB puncto A & concurrant crura CS, SC in D; applicetur dein concurſus crurum CB, SB reliquis punctis datis (2*n*) atque interea concurrant CS, SC in totidem punctis; per quæ & punctum D ex *Prop.* 26. ducatur Linea Ordinis (*n*) habens arcus (*n* — 1) concurrentes in D; & ſi concurſus crurum CS, SC ſemper ducatur per hanc Lineam, concurſus crurum CB, SB deſcribet Lineam quæſitam. Una duntaxat duci poteſt curva quæ huic Problemati ſatisfaciet, omniaque ejus puncta reliqua præter quatuor C, S, B, A ſimplicia eſſe neceſſe eſt per *Corol.* 5. *Lemmatis* 3.

Univerſaliter ducenda ſit Linea Ordinis (2*n*) per puncta C, S, B, A, &c. quot ad Lineam determinandam ſufficiunt; in C concurrere oporteat arcus (*t*) in S arcus (*g*) in B arcus (*t* + *g* — *n*) in A arcus (*r*) in E arcus (*m*) &c. Jungantur CS, SB, CB, & circa Polos C, S rotentur anguli BCS, BSC, ac concurſus crurum CB, SB applicetur punctis A, E dein reliquis datis dum interea concurrunt crura CS, SC in D, F & totidem aliis. Per quæ puncta & C, S, D, F ducatur Linea Ordinis (*n*) cujus puncta (*n*—*t*) ſint in C, (*n*—*g*) in S, puncta (*r*) in D & (*m*) in F, ſi modo iſthuc fieri poſſit; & ſi concurſus crurum CS, SC ducatur per hanc curvam concurſus crurum CB, SB deſcribet Lineam quæſitam.

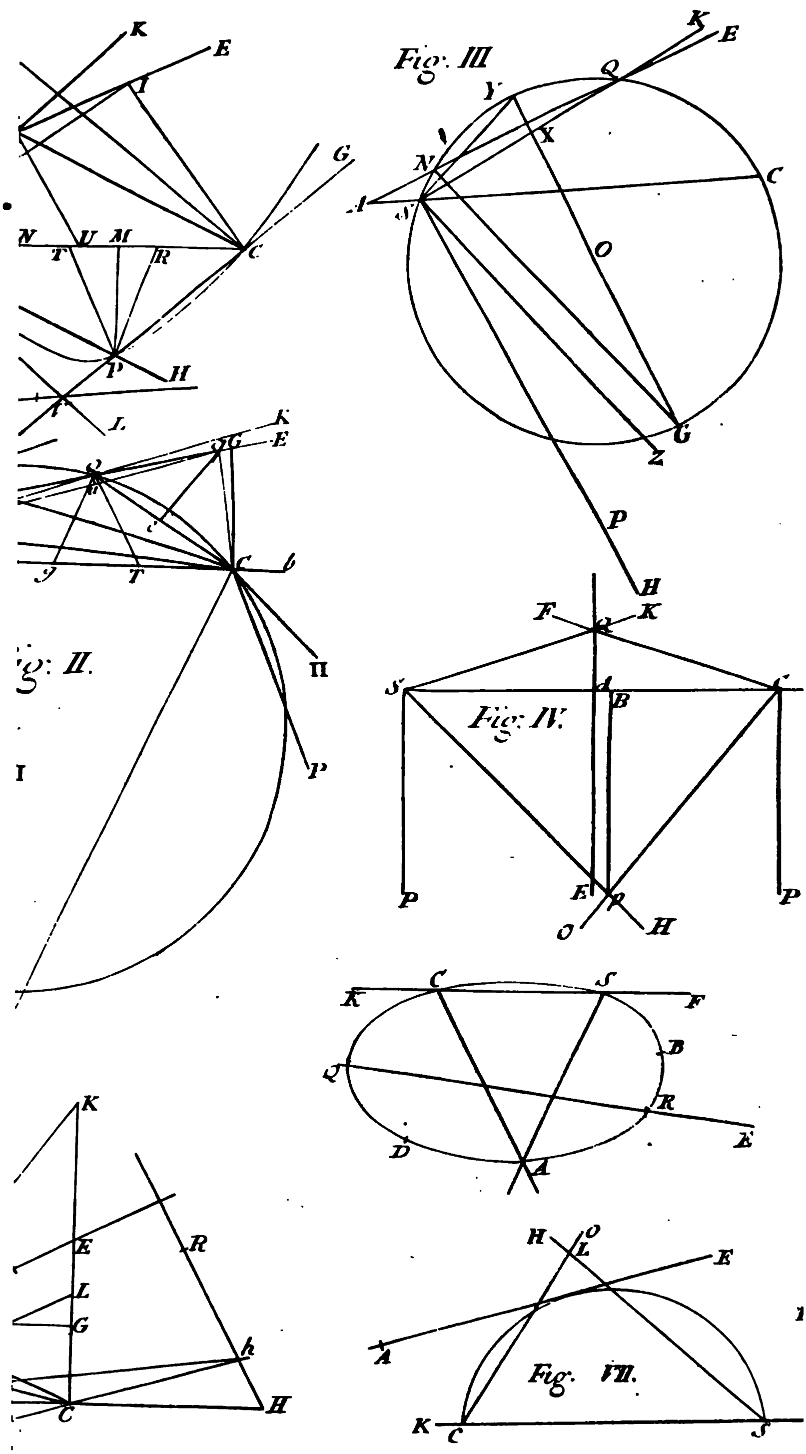
Ex *Propoſitionibus Partis* 1. Hujuſmodi quoque deſcriptiones deduci poſſunt, cum vero curvarum conſtructiones per puncta quævis propoſita, quas adhuc aſſequi potuimus, non ſint ſatis generales, his ulterius proſequendis non immorabimur.

FINIS.

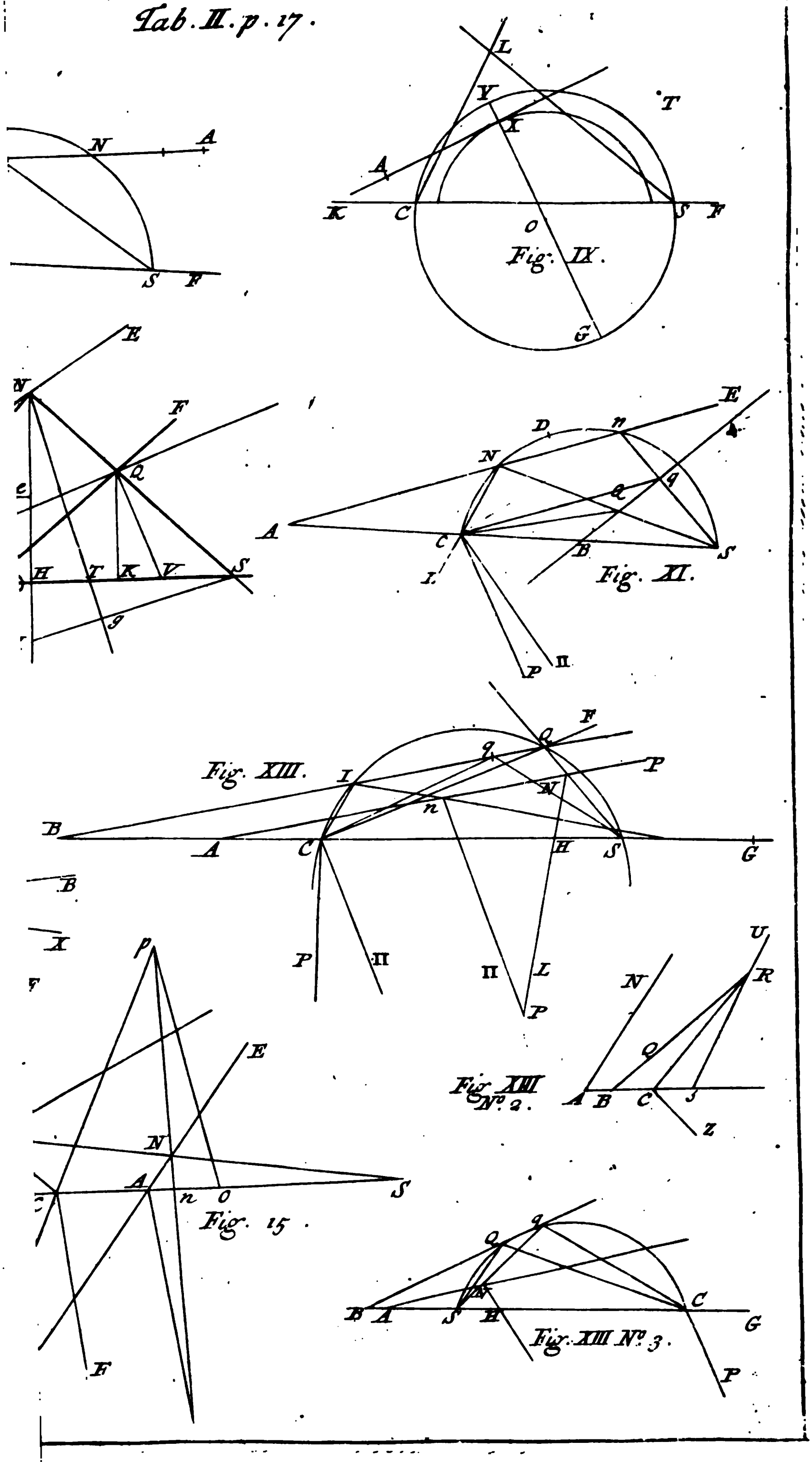
Corrigenda.

PAG. 1. lin. ult. pro QCA lege QCG. Pag. 4. lin. 21. lege *ex* S *in* AE. Pag. 5. lin. 28. pro QSK lege QSH. Pag. 8. lin. 14. pro LR lege HR. Lin. 15. pro EHR lege AHR. Pag. 22. lin. 2, 3 & 4. pro mk lege mk^2. Lin. 4. pro $i^2k - 2bk^2$ lege $i^2 - 2bk$ & poſt x^2 ſcribe *&c.* P. 23. Lin. 13. pro *Caſus* 1. lege *Caſus* 2. Lin. 22. lege *perpendiculares in* CS. Pag. 24. Lin. penult. pro $2c \pm a$ lege $2c \times c \pm a$. Pag. 27. Lin. 23. pro FB lege FD. Pag. 29. Lin. 23. pro QK lege SK. Pag. 30. Lin. 21. pro PHC lege QHC. Pag. 31. Lin. 6. pro CR lege CK. Pag. 36. Lin. 5. poſt DQ lege *Concurſus*. Pag. 38. Lin. 3. pro C & R lege C & Z. Pag. 48. Lin. 26. pro SNL lege CNL. Pag. 55. Lin. 6. pro CT lege KT. Lin. 8. pro b lege n. Pag. 63. Lin. 18. pro PTC lege PTB. Pag. 66. Lin. 20. pro QCP lege KCP. Pag. 69. Lin. 14. pro K lege R. Pag. 71. Lin. 6. pro 68 lege 72. Lin. 11. pro CQ lege BQ. Pag. 77. Lin. 9. lege *anguli* NS*n*, S*nr*, *nrt*, *&c. & anguli* MC*m*, C*ml*. Pag. 80. Lin. 29. lege (*Fig.* 1. *n.* 2.) Pag. 83. Lin. 27. pro AQB lege AQG. Pag. 91. Lin. 25. pro CSQ lege CQ. Pag. 96. Lin. 30. pro BLC lege BL*l*. Pag. 98. Lin. 12. pro SN lege FN. Pag. 99. Lin. 21. pro SLE lege PLE. Pag. 101. Lin. 6. pro $AC = c$, $AE = d$, lege $AC = c$, $CE = d$. Lin. 8. pro $+ by$ lege $- by$. Lin. 9. pro $x =$ lege $z =$. Pag. 108. Lin. 14. lege *ſummæ vel differentiæ*. Pag. 115. Lin. 3. ſcribe *Fig.* 28. Pag. 116. Lin. 12. pro *Prop.* 5. lege *Prop.* 16. Pag. 121. Lin. 28. pro 31 lege 29. Pag. 122. Lin. 17. pro L*o* lege *lo*.

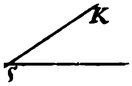
Tab. I. p. 9.



Tab. II. p. 17.



N A S F
L Y X T A K C O S F G
Fig. IX.
E N F Q e H T K V S g
E D n N q Q A C B S I P Π
Fig. XI.
F Q q P N Fig. XIII. I n B A C H S G P Π Π L P
B X
U N R Q A B C s Z
Fig. XIII N°. 2.
p E N A n o S F
Fig. 15.
q Q N B A S H C G P
Fig. XIII N°. 3.



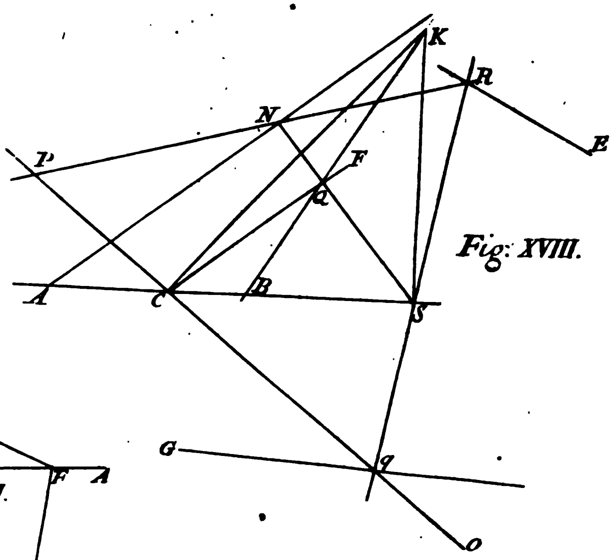


Fig: XVIII.

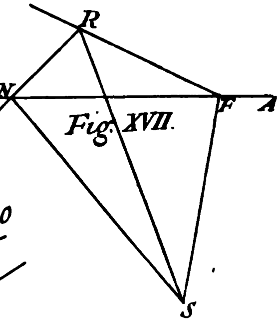


Fig: XVII.



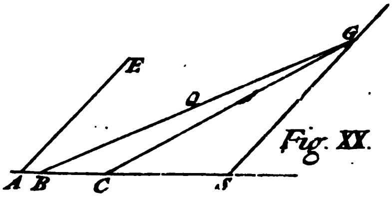
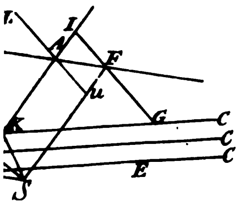


Fig: XX.



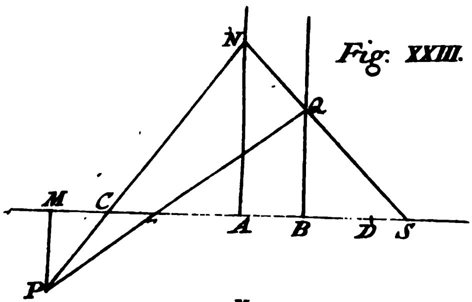
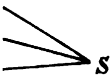


Fig: XXIII.





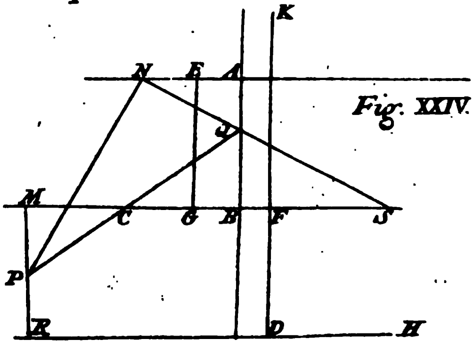
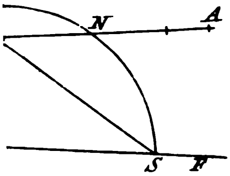


Fig: XXIV.

Tab. II. p. 17.



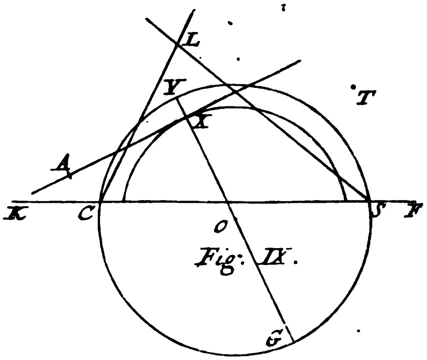
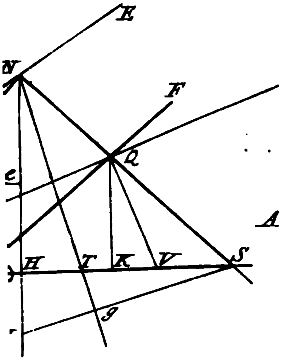


Fig. IX.



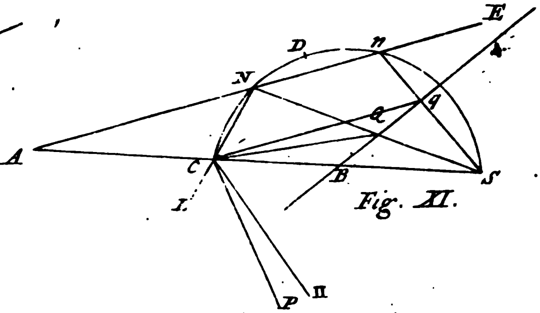


Fig. XI.

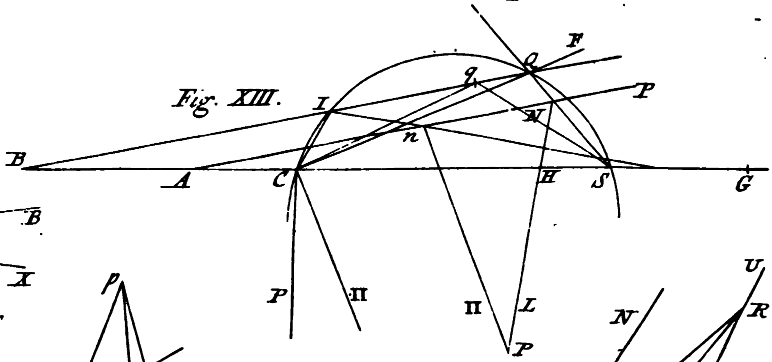


Fig. XIII.

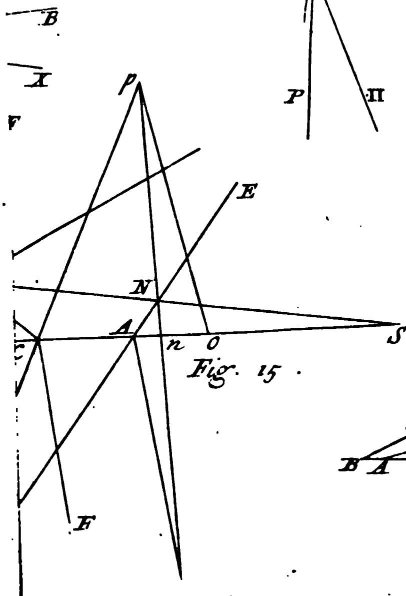


Fig. 15.

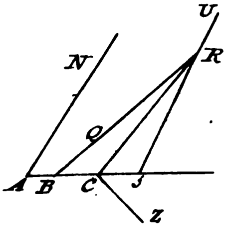


Fig. XIII N°. 2.

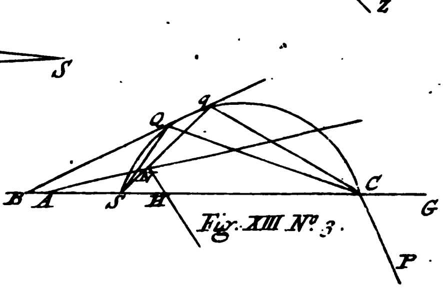
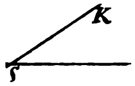


Fig. XIII N°. 3.



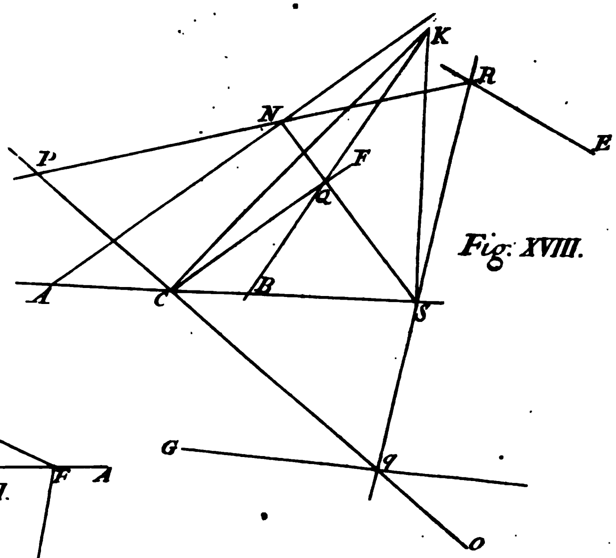


Fig. XVIII.

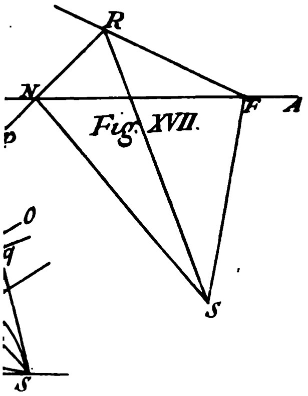


Fig. XVII.

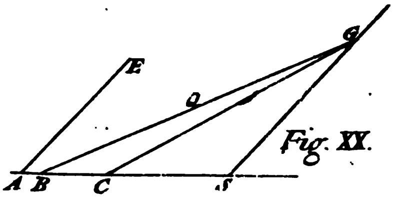
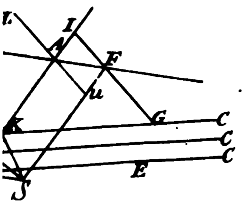


Fig. XX.



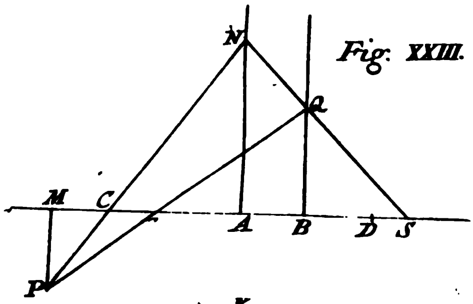
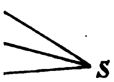


Fig. XXIII.





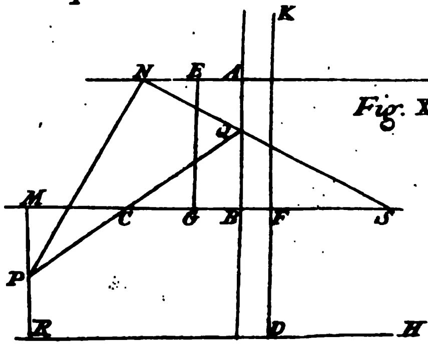
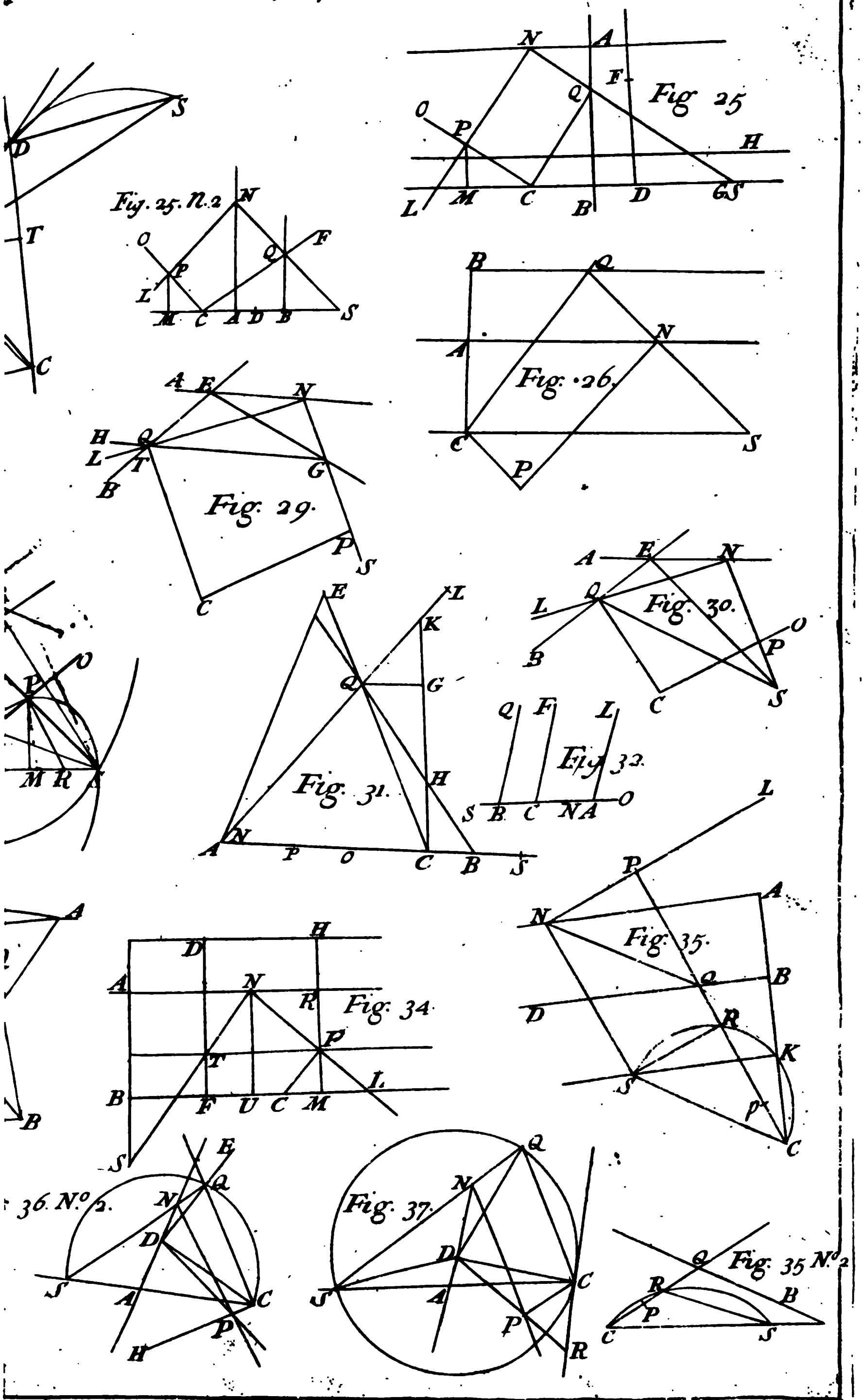
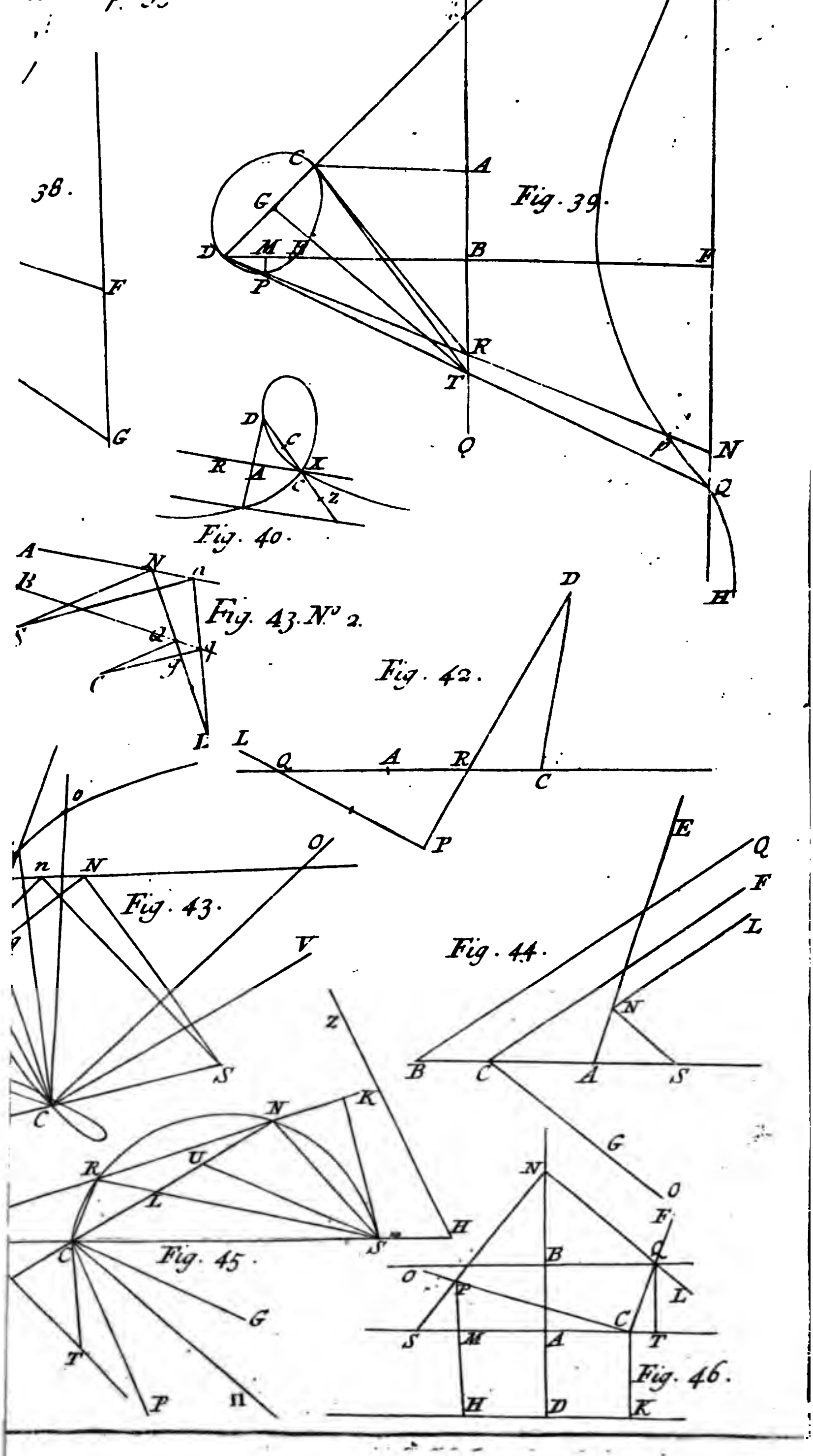


Fig. XXIV.

Tab. IV. p. 27.


Fig. 25
Fig. 25. N. 2
Fig. 26
Fig. 29
Fig. 30
Fig. 31
Fig. 32
Fig. 34
Fig. 35
Fig. 35 N.° 2
36. N.° 2.
Fig. 37


38.
Fig. 39.
Fig. 40.
Fig. 43. N° 2.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46.

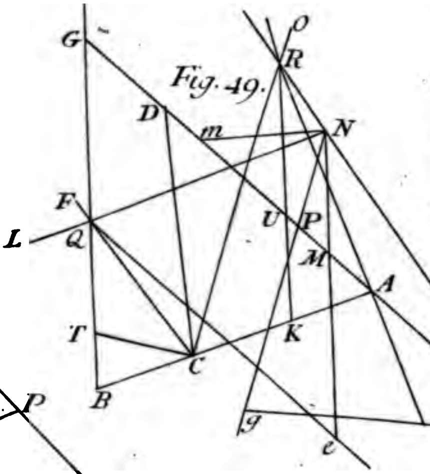


Fig. 49.

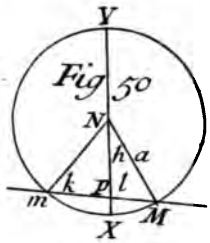


Fig. 50

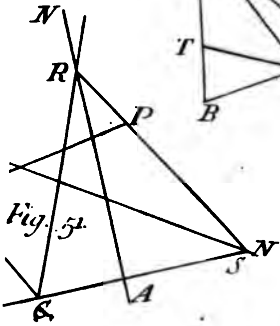


Fig. 51.

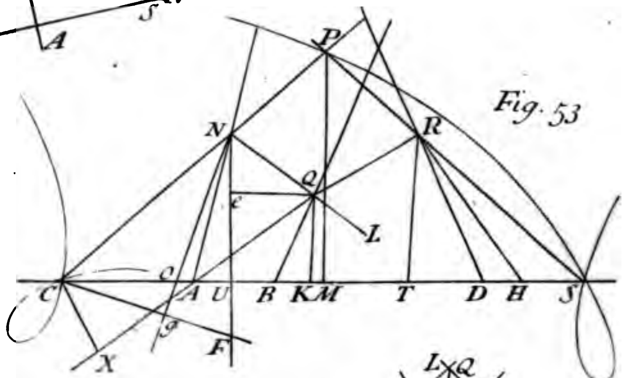


Fig. 53

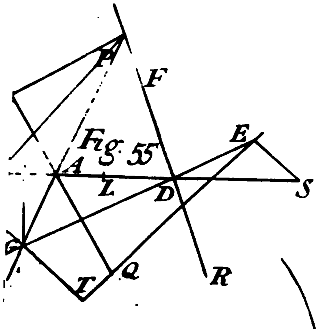


Fig. 55.

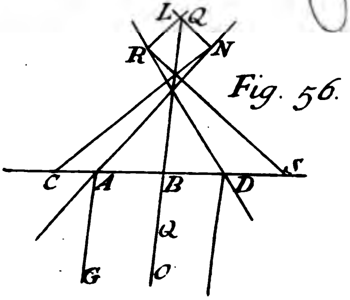


Fig. 56.

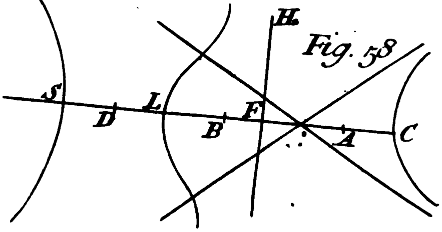
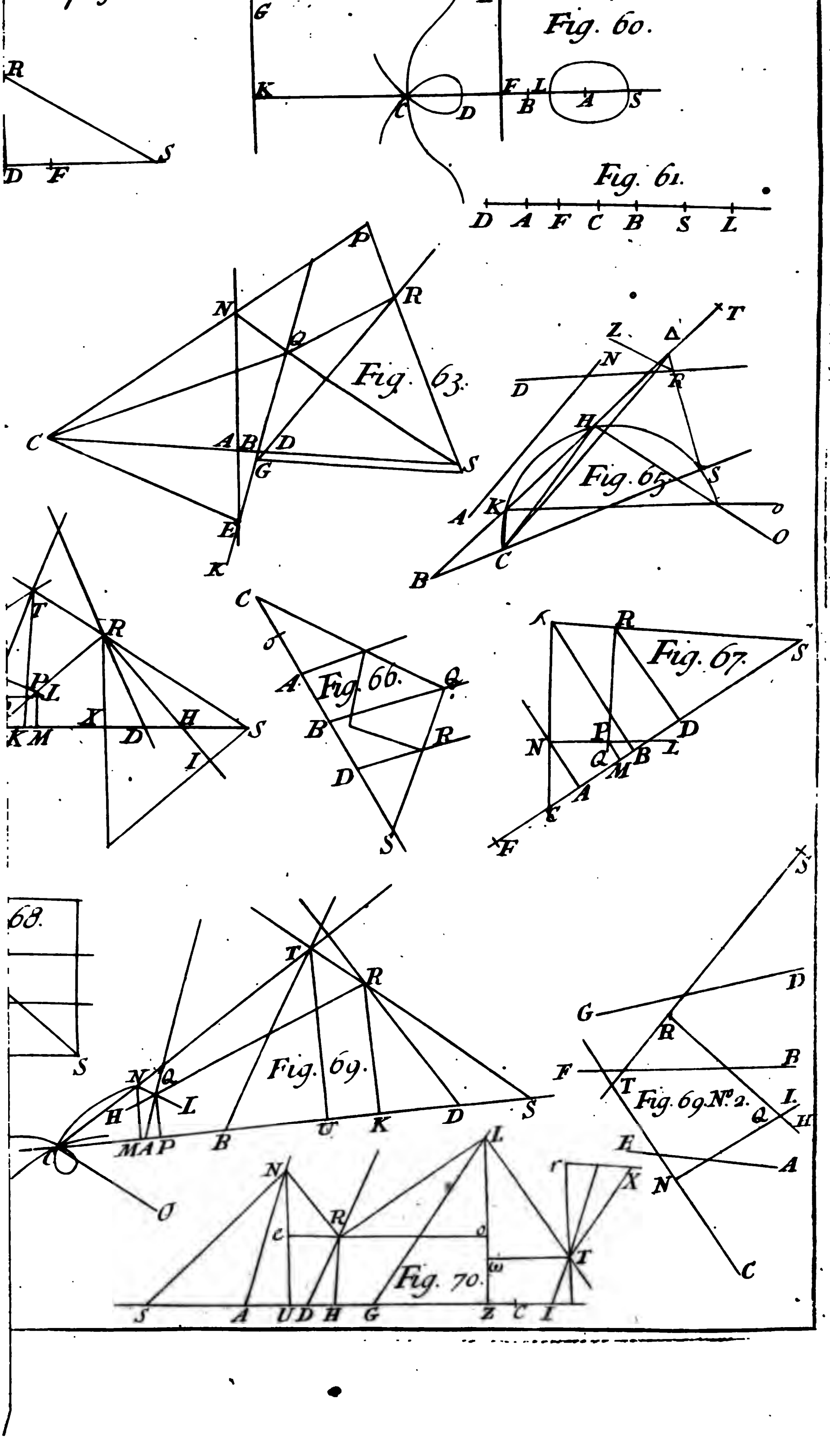
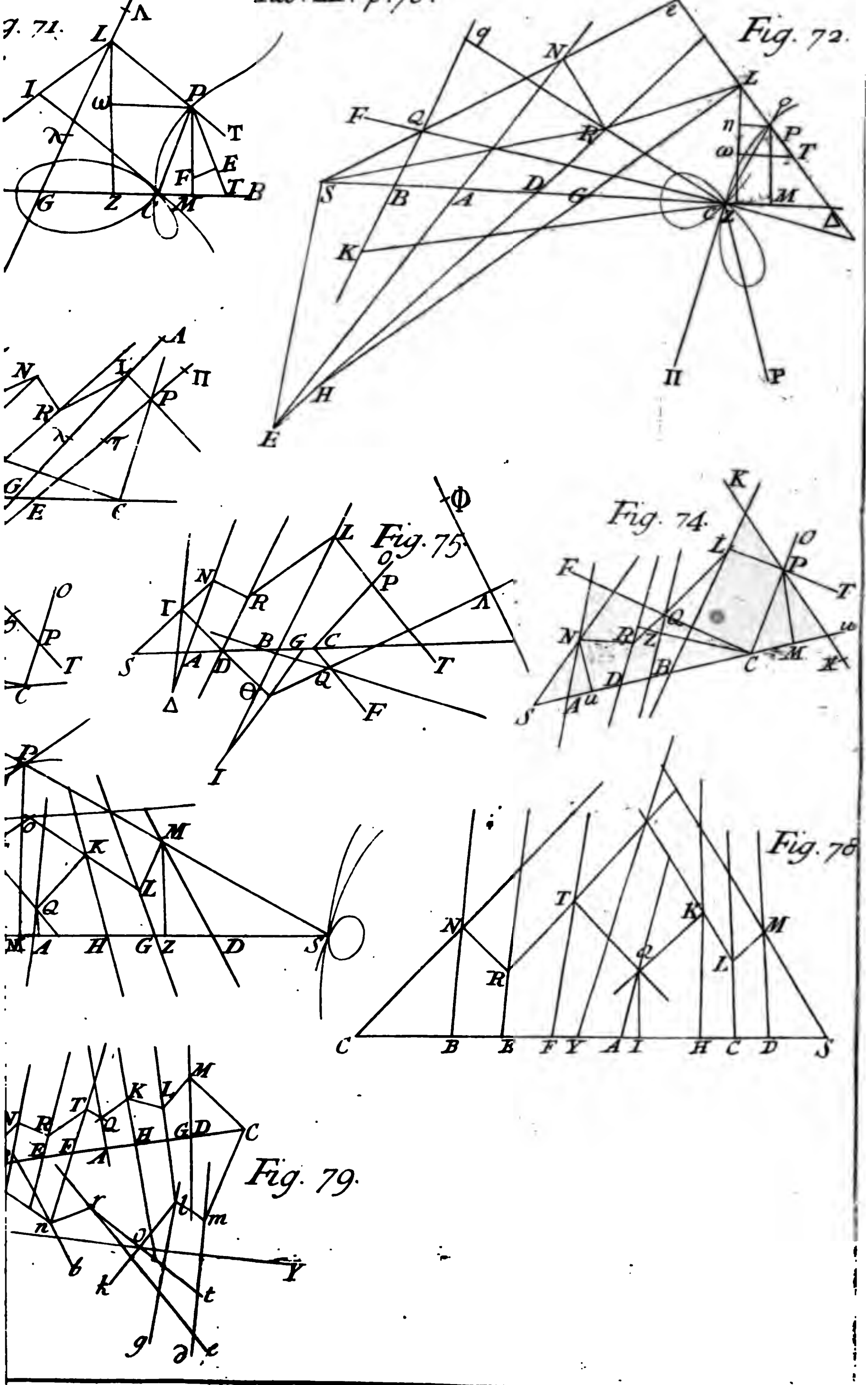


Fig. 58

. VII. p. 56.


Fig. 60.
G H K C D F L B A S
Fig. 61.
D A F C B S L
Fig. 63.
P N R Q C A B D G S E K
Fig. 65.
T Z Δ N D R H S A K C B O
T R L X D H S K M I
Fig. 66.
C A Q B R D S
Fig. 67.
R S P N D Q B L M A F
68.
S
Fig. 69.
T R N Q H L M A P B U K D S C O
Fig. 69. N.° 2.
S D G R F B T I Q H E A N C
Fig. 70.
L N R X T S A U D H G Z C I

Tab. VIII. p. 78.



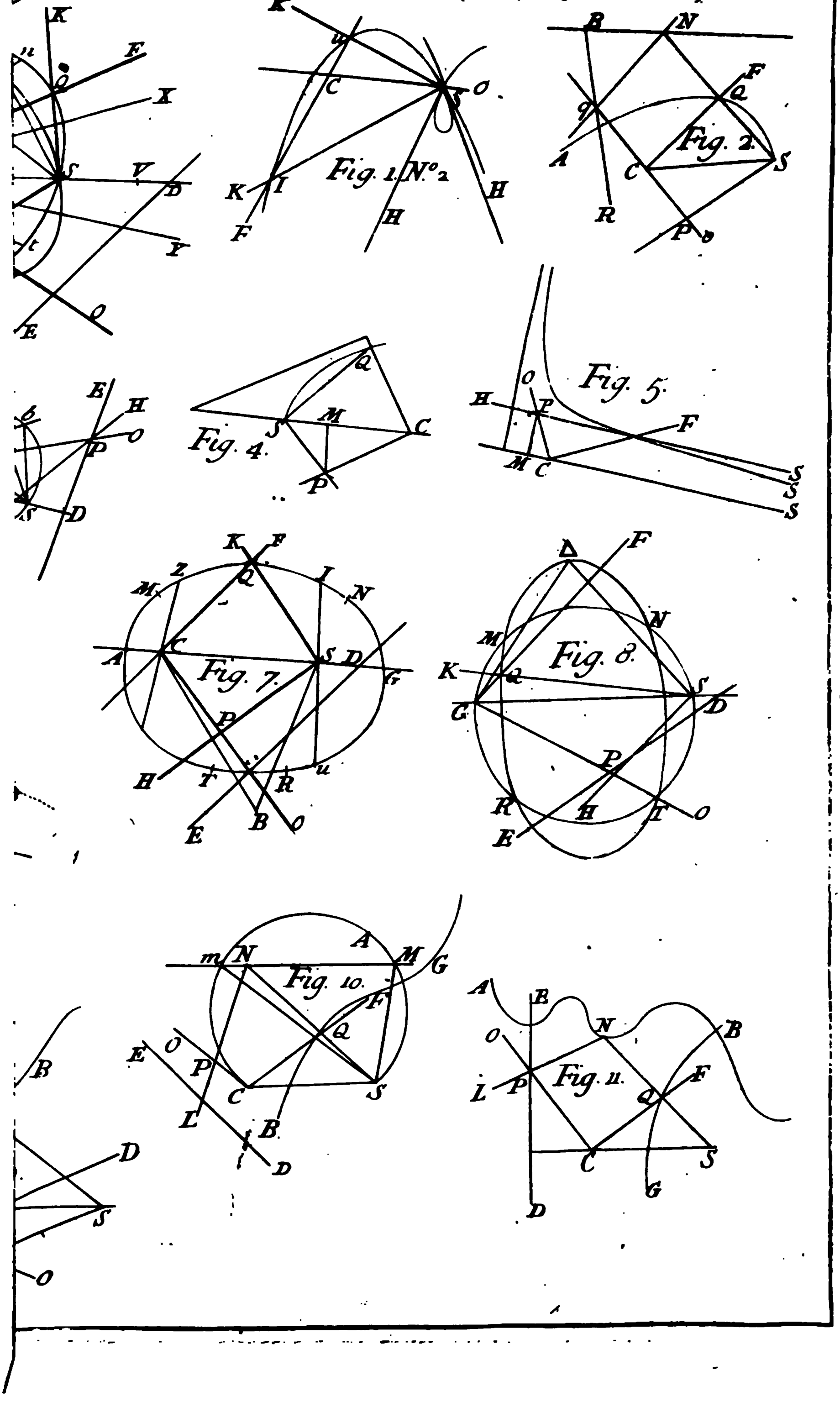
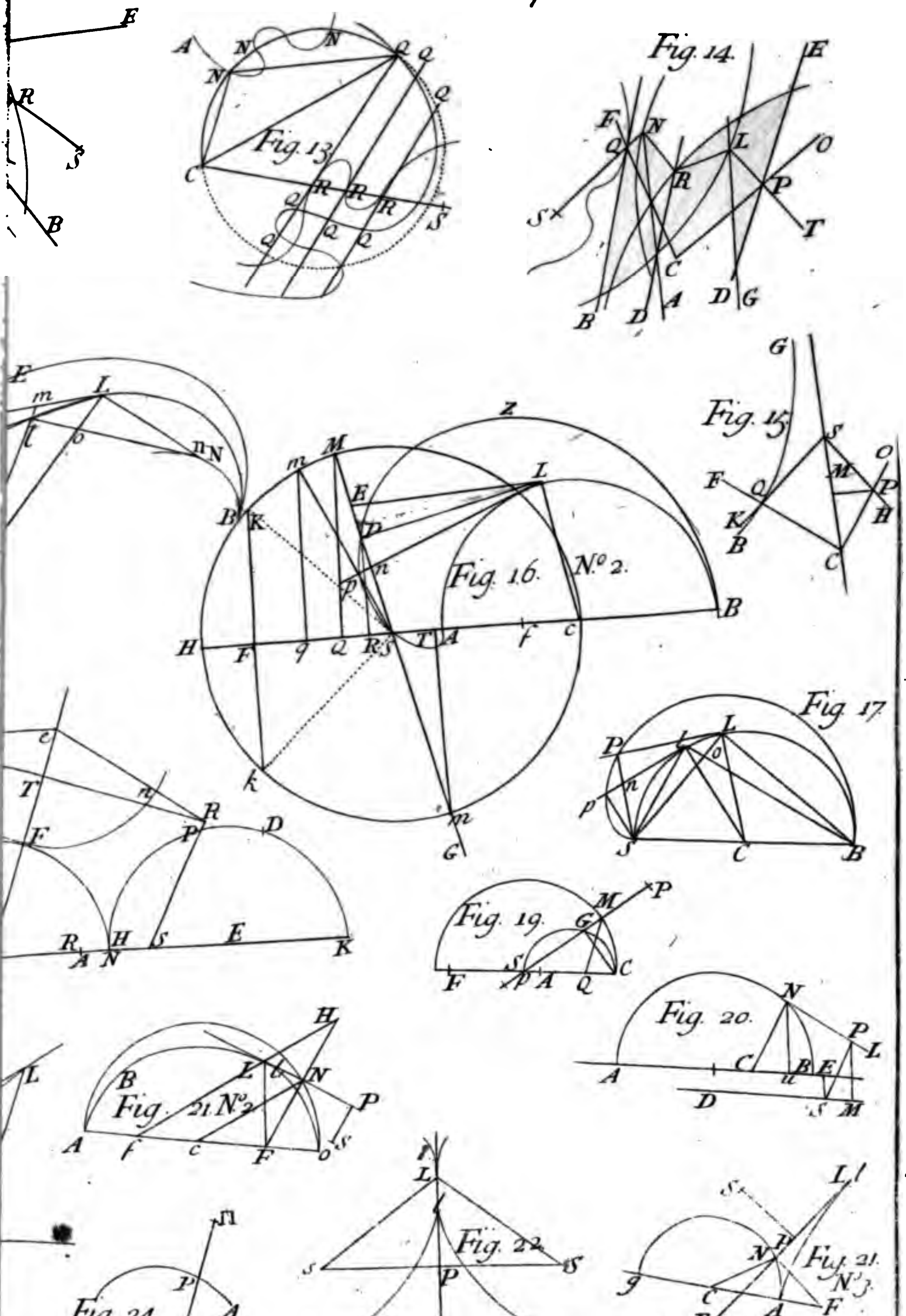
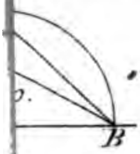


Fig. 1. N°. 2
Fig. 2
Fig. 4.
Fig. 5
Fig. 7.
Fig. 8.
Fig. 10.
Fig. 11.


Fig. 13
Fig. 14
Fig. 15
Fig. 16
N.º 2.
Fig. 17
Fig. 19
Fig. 20
Fig. 21 N.º 2
Fig. 21 N.º 3
Fig. 22
Fig. 24



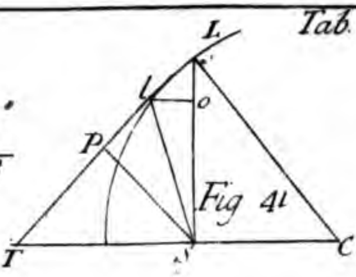


Fig 41

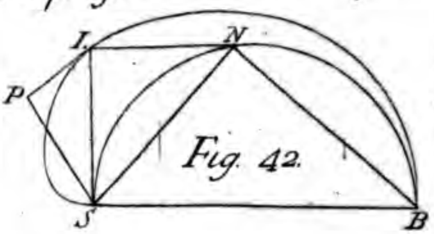


Fig. 42.

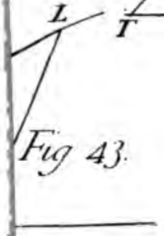


Fig 43.

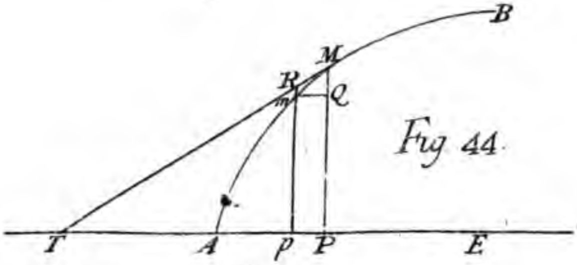
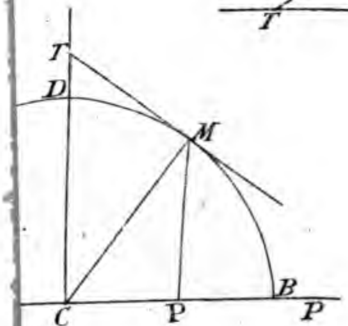


Fig 44.



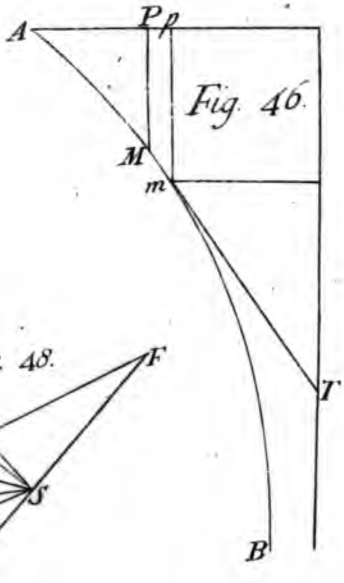


Fig. 46.

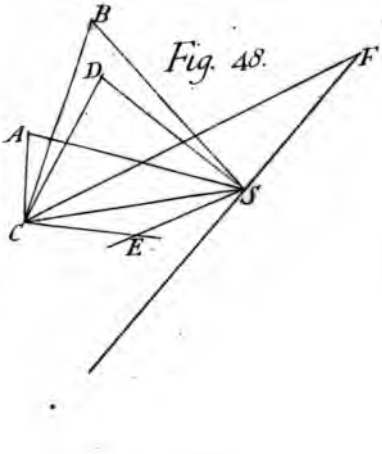
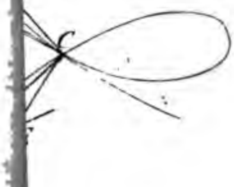


Fig. 48.



6/V Bzm